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ELEMENTARY MECHANICS

INCLUDING

HYDROSTATICS AND PNEUMATICS

BY

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New Edition

COMPLETELY REVISED BY THE AUTHOR AND BY

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PREFACE TO THE 1896 EDITION.

THE present book aims at giving a clear knowledge of the principles of the subject, in as elementary and even popular a manner as is consistent with careful accuracy, and without assuming any mathematical knowledge beyond the most rudimentary algebra. At the same time it is hoped that students who use this manual will be able to master the elements of the science in such a way that they may rise from it to more advanced treatises, not only without having anything to unlearn, but with a very sound knowledge of principles. Copious illustrations and explanations have been inserted, and the needs of students who are without the aid of a teacher have been kept steadily in view.

The subject is treated as an introduction to Physics, and its laws are deduced from the first principles of familiar experience rather than from special experiment. Experiments in Mechanics have a subordinate though most useful part in illustrating and emphasising the facts, but the author has no faith in making the establishment of principles depend on special experiments. So also in Geometry: drawing, measuring, and weighing may well be used for purposes of instruction and illustration, but propositions should be otherwise proved.

The early examples at the ends of the chapters are typical ones, and are intended not only to be worked without looking

at the answers, but also to be read almost as part of the book, because they frequently direct attention to important details. A large number of examples for practice have now been added to these, and the text has been thoroughly revised. In this work, as stated on the title-page, the author has had the collaboration of his brother.

The statements made in a book should be carefully criticised, and not taken for granted; and all kinds of special cases should be thought of or tried, to see if an exception cannot be found. *It is by thinking one's self on a subject that it becomes really known to one's self; it will never be really known if we only try to understand and remember what the book says.*

The author thinks that students will derive benefit from referring to Part I. of Deschanel's *Natural Philosophy*, translated by Dr Everett, as a supplementary well-illustrated work introductory to general Physics, and reference is accordingly made to it or to the corresponding portion of Ganot's *Physics* for details which would unduly swell the size of the present book. From the more engineering side, Professor Perry's *Practical Mechanics* is also to be recommended.

The book, as now revised, is intended to be not only an easy introduction to the subject, but, as far as it goes, a philosophical work. If at any place it is unable to stand the test of hostile criticism, the failure is a defect which the author will gladly utilise the aid of the critic to remove. From friendly critics he has already received several welcome minor corrections.

OLIVER J. LODGE.

SUGGESTIONS FOR READING.

BEGINNERS are recommended to omit the following sections on a first reading: 17, 18, 31, 40–43, 53, 55, 56, 76, 80–82, 104, 105, 123; and then to return and read the omitted portions together, and finally to read the whole book carefully through without omitting anything. Students preparing only for London University matriculation, or for the elementary stage of the Science and Art Department, may with safety omit any of the above sections over which they experience much difficulty until the examination is over. The introduction being harder than many other parts of the book, its complete reading may be deferred. It is inadvisable to begin the study of either Mechanics or Physics without a knowledge of the Greek alphabet.

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ELEMENTARY MECHANICS.

INTRODUCTION.

ON FORCE.

1. **Physics** is the comprehensive science which deals with the general relation and properties of the three fundamental facts or phenomena—Space, Time, and Matter.

Mechanics is the foundation of Physics, and deals with the simplest and most direct relations among these same phenomena.

(There are other branches of Natural Philosophy, such as Chemistry, which discusses particularly the properties whereby forms of matter differ from each other, and Astronomy, which deals with the motions and constitutions of large and distant masses of matter.)

Metaphysics, on the other hand, attempts to solve problems as to the ultimate nature of the above phenomena, seeking to express them in terms of mind and consciousness, or *vice versâ*; and it also considers how far the things called space, time, and matter really exist. Physics silently accepts their existence, and seeks to express all their properties and relations in the simplest terms. The science of Mechanics or Dynamics embraces that part of Physics in which this attempt has been so far successful. This is the scientific use of the term Mechanics; it used to mean chiefly the science of Machines, and this is still a part of the subject; usually now this part is specially distinguished as *Applied Mechanics*, and the more general aspect of the science itself is

called **Dynamics**, which signifies a treatment of the action of the fact or conception which links together the three fundamental phenomena already specified—namely, the very important fact or conception of 'Force.'

2. By the term **Force** we are to understand muscular exertion, and whatever else is capable of producing the same effects.

Muscular action impeded gives us our primitive idea of force; our sense of muscular exertion itself is a primary one for which we have special nerves, and it is not resolved into anything simpler. When any inanimate agent produces an effect on bodies exactly similar to that which would be produced by muscular exertion on the part of an animal, it also is said to exert force. Thus, a steam-engine exerts force when propelling a carriage, or pumping water, or turning a mill; gunpowder exerts force on a cannon-ball during the time the ball is passing from the breech to the muzzle of the gun. But in order that an agent may exert force it must meet with some resistance; in other words, force is always the mutual action of *two* bodies against one another, and the amount of the force is precisely equal to the amount of resistance. Thus, a flying meteor or a cannon-ball is not exerting force (except, indeed, on the earth, by reason of the fact of gravitation) unless it meets with some resistance: but if the air rub against it and resist its motion, it will exert a force against the air; and when it strikes a target, it meets a very great resistance, and therefore exerts a very great force, possibly smashing the target. A running stream exerts very little force unless it meets with an obstacle; but if you resist its motion with your hand, it will press against your hand; or if you dip in the vanes of a water-wheel, it may force the wheel round.

3. **Stress.**—Forces, then, always occur in pairs, constituting a mutual action, a pulling or pushing between *two* bodies, and the action (pull or push) of the one on the other

is always precisely equal to the reaction (pull or push back) of the other on the one. In other words, action and reaction are equal and opposite. This pair of forces which always go together it is convenient to have a name for, and it is called a *stress*. It is a *tension* if the forces are acting away from each other, and a *pressure* if they are acting towards each other. If the stress be directly between two different bodies, it is always a pressure; if between different parts of the same body, as at any section of a rod, it may be either a pressure or a tension, and is in either case called an internal stress. (The effect which a stress produces in an ordinary solid before rupture is called a *strain*; see sect. 5.)

There are indirect actions between bodies, such as *Gravitation*, *Cohesion*, *Magnetic and Electric Forces*, &c., which are not yet thoroughly understood, but which probably arise from internal stresses in some energetic connecting medium which thus exerts equal pressures on both bodies, forcing them towards or away from each other, as the case may be. So far as the bodies are concerned, therefore, these forces may be classed under the head of *Pressures*, not between the two bodies, but between the medium and each body. When the earth is one body and a stone the other, the gravitational pressure which is driving them together is commonly called the 'weight' of the stone. The peculiarity of gravitational pressure is that it acts on every atom of both bodies throughout their entire mass. Where two contiguous particles of the same body are being considered, the pressure which holds them together is called *cohesion*: and it is the existence of this remarkable molecular stress which permits the possibility of *Tension* of any kind in material bodies.

It is often convenient to isolate one of the components of a stress between a pair of bodies, and to consider only the force acting on one body; but we only do so by *attending* to this one and neglecting the other component, which always necessarily

exists and is acting on the other body of the pair. Moreover, which of the forces we choose to call the direct action, and which the reaction, is merely a matter of convenience; but it will be obviously convenient to speak of that component which acts on the piece of matter we are dealing with as *the* force, or the *action* of the other piece of matter, while that component which affects this other piece of matter will of course be the *reaction* of the first piece on it. An absurd puzzle is sometimes made out of the fact that a cart always pulls back the horse with precisely the same force as the horse pulls forward the cart. It is asked, 'How, if that be so, can they ever start?' The puzzle can only be felt by those who forget that each force acts on a different body. In the simplest case, there is no equilibrium or balance of forces acting on the cart; there is only one force acting on the cart—namely, the pull of the horse, and this force is quite unbalanced; the reaction or pull back of the cart does not act on the *cart*, but on another body altogether—namely, the horse. It needs therefore no knowledge of mechanics to see clearly the way out of this puzzle; it needs only a little thought and some common-sense. To explain why a horse or a steam-engine is able to expert force at all—that is, to explain how the system of horse-and-cart is able to progress—is more complicated.

We have spoken of force as exerted *by matter*. Of inert matter this is hardly correct. Matter does not *of itself* exert force; it must be set in motion, or have some other form of **Energy** conferred upon it, before it can exert force.

Remembering this, however, we shall do no harm by habitually using the convenient phrase, 'the force exerted by such and such a body.' Even in the case of the indirect actions between two bodies, such as gravitation, &c., above mentioned, it is customary to talk of the force as if it were exerted by one body on the other, although, strictly speaking, the forces are between each body and the connecting medium. Thus we say that the weight of a stone is due to the pull of the earth upon it; but, since the action is really a stress in the surrounding medium, it follows that a stone exerts precisely the same force on the earth as the earth exerts on the stone.

4. **Equilibrium.**—A book lying on a table is at rest. Why? Not because no force is acting on it, for the earth is pulling it; but because another and equal force is also acting on it in the opposite direction—namely, the resistance of the table. This is the condition of all bodies at rest near the surface of the earth; they are subject to two or more forces which neutralise each other as far as *motion* is concerned, though they do not neutralise each other as regards *strain*. Indeed, when we wish to produce strain and not motion, we must subject the body to the action of two equal opposite forces; for example, if you want to tear a piece of paper, or break a string, or stretch a piece of elastic, or crack a nut, it is no use pulling or pushing at one side only; you must apply a force to both ends or sides—that is, you must apply a tension or a pressure; you must subject the body to an internal stress.

Strictly speaking, *motion* appears to be the normal condition of matter at present; all known bodies are moving through space with considerable speed, and no such thing as *absolute rest* is known. We can, therefore, only consider the motion of bodies relative to some body regarded for the time being as fixed. It is generally convenient, in mechanics proper, to consider the earth as a body at rest, and to leave the study of the motion of it, and of the group of bodies to which it belongs, to Astronomy, which is really a branch of mechanics in a wide sense. It may sometimes be convenient to consider the earth to be moving through space in any direction desired. It is also often advantageous to consider the motions of two bodies, or parts of a body, as if one of the two were fixed, although both or either may be moving relatively to the earth. For example, on board ship, one generally considers the motions of the various people and movables with regard to the ship, and not with regard to the water or dry land; and in applied mechanics the study of the relative motions of the various parts of a machine is very important.

5. The effects of force on matter are:*

* Whatever other effects of force there may appear to be, are studied under *Physics*, and physicists are hoping to reduce all of them ultimately to the above two forms. Hence Physics is constantly tending to become more and more mechanical.

A. Change of motion, which is called **Acceleration.**

B. Change of size or shape, which is called **Strain** or deformation.

If only one force acts on a body, it must produce the effect A, and it may produce B also. If two or more forces act in different directions on different parts of a body which is not absolutely stiff and rigid, they must produce B, and they may produce A also.

6. The two kinds of effect, A and B, are distinct; and each would furnish a *measure* of force.

A force may be measured by the amount of motion it can produce in a given piece of matter in a given time; and this is the measure we shall mostly use.

Or a force may be measured by the amount of strain it can produce in a certain piece of matter: the amount it can bend a certain spring, for instance, as in a dynamometer; or the amount it can twist a certain wire, as in a torsion or spiral spring balance. If we are not concerned with *measuring* forces absolutely, but merely wish to *compare* two forces, we may of course simply balance them one against the other, as is done in a balance or steelyard.

Of the two classes of effect, A and B, A is much the simpler, and constitutes the branch of mechanics of which a portion is studied in an elementary course; it is the only branch suited to elementary exposition such as the present. But before proceeding to our actual subject, the motive effect of force (called *Dynamics*, from δύνᾰμις, force), it is convenient to study motion itself a little in the abstract, and without reference to either force or matter. (The subject of abstract motion is called *Kinematics*, from κίνημα, motion.) We may conceive a geometrical point or surface moving about in all sorts of ways without troubling ourselves with the cause of the motion, and the propositions which we so discover will be useful when we come to the motion of an actual piece of matter under the influence of a force.

CHAPTER I.

ON MOTION (*Kinematics*).

I. MOTION OF A POINT (TRANSLATION).

(*a*) *Rectilinear Motion.*

7. A body is said to *move* when it is in different positions at different times. This is to be regarded as the essential characteristic of motion—it involves a reference to both space and time. Geometry deals with space alone. Kinematics deals with both time and space.

Now motion has two primary properties to be studied—Speed and Direction; both of which are sometimes held to be included under the one name, Velocity. Let us take them in order.

When a body moves over equal spaces in equal times, its motion is said to be *uniform*, or its speed is said to be *constant*.* For instance, the tip of the hand of a clock has such a motion as regards speed, in spite of the fact that its direction of motion is constantly changing. The apparent motion of a fixed star across the field of a telescope is another instance of uniform motion.

When a body moves over *un*equal spaces in equal times, its velocity is said to be variable. As an example of *variable* velocity, we may take the case of a falling stone, which moves quicker and quicker as it descends; or of a stone thrown upward, which has a decreasing velocity till it

* It is probable that our idea of motion (that is, of free muscular action) precedes and suggests our idea of time; and that our notion of *equal intervals* of time depends on our recognition of *uniform motion*. Every measurer of time is simply a uniformly moving body. The most uniformly moving body we know is the earth, which rotates on its axis in a period of always the same duration; this period is taken as our fundamental unit of time, and the $\frac{1}{86400}$th part of it is called a *second* of 'mean solar time,' and is used as the practical unit.

reaches its highest point; or of the bob of a pendulum, which has a velocity alternately increasing and decreasing, as well as changing in direction. To begin with, we shall consider motion in a straight line only—that is, with constant direction.

8. **Velocity** is defined as the rate of motion of a body. When uniform, it is measured by the distance travelled, divided by the time taken in the journey; when variable, its average value is measured in the same way. Thus, if a point move over a distance s in a time t, its velocity is s/t;

$$\text{or } V = \frac{s}{t}.$$

For example, if a train goes 80 miles in 4 hours, its average speed is 20 miles an hour. This expression may be put in the form of a fraction, and may be written

$$\frac{80 \text{ miles}}{4 \text{ hours}} = \frac{20 \text{ miles}}{1 \text{ hour}} = 20 \times \frac{1 \text{ mile}}{1 \text{ hour}};$$

the last fraction denoting a velocity of 1 mile per hour. Similarly, $\frac{1 \text{ foot}}{1 \text{ second}}$, or, shortly, $\frac{\text{ft.}}{\text{sec.}}$, denotes a speed of 1 foot per second, which we may consider as a sort of standard British unit of speed, suited to the majority of problems with which we shall have to deal. The speed of the above train may if we please be reduced to feet per second, as follows:

$$\frac{80 \text{ miles}}{4 \text{ hours}} = \frac{80 \times 1760 \times 3 \text{ feet}}{4 \times 60 \times 60 \text{ seconds}} = \frac{88}{3} \text{ feet per second.}$$

The mode of dealing with units indicated at full length in this extremely simple example will be found of considerable service in more complex cases.

Note that a velocity is *length per time*, and that it is not correct to speak of a velocity of so many *feet*. We speak of a *length* of so many feet—or a *time* of so many seconds—but a *velocity* of so many *feet per second.*

The **unit** of velocity is of course $\frac{\text{unit of length}}{\text{unit of time}}$; that is, $\frac{1 \text{ foot}}{1 \text{ second}}$ or $\frac{1 \text{ centimetre}}{1 \text{ second}}$ (read 1 foot, or 1 centimetre, per second). Strictly speaking, it is incorrect to speak of a velocity 6 simply, but it is sometimes done when the particular unit of velocity is specified by the context.

9. The above measure of velocity as the ratio of s to t is independent of the size of s and t, so that it remains perfectly true when s and t are very small. Thus in the case of a body moving uniformly 6 feet every second, its velocity may be written either $\frac{6}{1}$ or $\frac{1}{\frac{1}{6}}$ or $\frac{\frac{1}{1000}}{\frac{1}{6000}}$; and any of these fractions represents its velocity *equally well so long as it be uniform*. But if the velocity were variable, the body might still go 6 feet in a second, so that its *average* velocity would still be 6; but its *actual velocity at each instant* might take all kinds of values, some greater and some less. Thus a train which had gone from London to York, 200 miles, in 5 hours, would have had an average speed of 40 miles an hour; but its actual speed would have varied greatly; sometimes rising to 60 perhaps, sometimes falling to 0, as at a station. The whole distance travelled, divided by the whole time taken, will always give us the *average* velocity for that distance; and in the case of *uniform* motion, the average velocity coincides with the *actual* velocity at each instant. But to get information on the actual velocity, at any one place, of a thing whose speed varies continually, it is necessary to suppose a *small* distance taken at that place, and divided by the time taken to traverse it. The smaller the distance taken, the less possibility is there of variation, and the more exact will the specification be; hence the actual velocity of any moving body at a given instant is the infinitely small distance then being described divided by the infinitely small time required for the purpose.

(The facts are often expressed in a form which appears more simple, but which involves less important ideas—namely: Uniform velocity is measured by the space described in unit time. Variable velocity, by the space which *would* be described in a unit of time if at the given moment the velocity were to cease to vary.)

So then, using little v to stand for actual velocity at any instant, $v=\frac{s}{t}$ is true when s and t are small; but, using big V to stand for average velocity throughout any time, $V=\frac{s}{t}$ is always true unconditionally.

EXAMPLES—I.

(1) A train is travelling at the rate of 25 miles an hour. How long will it take to travel between two telegraph poles 100 yards apart?

There are usually about 24 telegraph-poles to the mile, hence the speed of a train may be roughly estimated by a traveller. It may be shown in fact that in that case the speed of the train, in miles per hour, is 150 divided by the number of seconds taken to travel between two poles. Because if

$$\frac{x \text{ m.}}{1 \text{ hour}}=\frac{1 \text{ tel. pole}}{s \text{ seconds}},$$

it follows that

$$x=\frac{1 \text{ hour}}{s \text{ seconds}}\times\frac{1 \text{ tel. pole}}{1 \text{ mile}}=\frac{3600}{24\,s}=\frac{150}{s}.$$

Another way of putting the result is to say that the speed, in miles an hour, is $2\frac{1}{2}$ times the number of telegraph poles passed per minute.

(2) A man walking from A to B at $3\frac{1}{2}$ miles an hour arrives at B in 1 hour 35 minutes. A cyclist starting from A an hour later arrives at B at the same time; at what rate per hour was he travelling?

(3) If 100 inches = 254 centimetres, how many centimetres per second is equivalent to 3 miles an hour?

(4) A man walks at the rate of 2 yards a second. What is his speed in miles per hour?

The result is a convenient fact to remember: 2 yards a second is 4 miles an hour, roughly.

(5) How many feet does the tip of the minute hand of a clock travel in 24 hours if it is 4 feet long? How much does it

move for each complete swing of its pendulum if it ticks 40 to the minute?

(6) If a snail crawl at the rate of $\frac{1}{4}$ inch a second, how far will it go in an hour?

(7) How long would a train take to go 100 yards at the rate of 20 miles an hour?

(8) With what velocity must I walk in order to go half a mile in five minutes?

ANSWERS TO I.

(Worked in full to show the mode of dealing with units.)

(1) $$t=\frac{s}{v}=\frac{\overset{3\times 30}{\cancel{100}\ \cancel{\text{yards}}\times\cancel{60}\times\cancel{60}}\ \text{seconds}}{\underset{\underset{11}{\cancel{440}}}{\cancel{25}\times\cancel{1760}}\ \cancel{\text{yards}}}=\frac{90}{11}\text{ seconds}=8\tfrac{2}{11}\text{ seconds.}$$

(2) $$s=\frac{3\frac{1}{2}\text{ miles}}{1\text{ hour}}\times 95\text{ minutes}$$

$$v=s\div t=\frac{3\frac{1}{2}\text{ miles}\times 95\ \cancel{\text{minutes}}}{1\text{ hour}}\div 35\ \cancel{\text{minutes}}$$

$$=\frac{3\frac{1}{2}\times\overset{19}{\cancel{95}}}{\underset{7}{\cancel{35}}}\text{ miles per hour}$$

$$=9\tfrac{1}{2}\text{ miles per hour.}$$

(3)
$$\begin{array}{r}5280\\254\\\hline 105|60\\26|400\\2|1120\\\hline 134|\end{array}\qquad \frac{3\text{ miles}}{1\text{ hour}}=\frac{3\times 1760\times 36\text{ inches}}{60\times 60\text{ seconds}}$$

$$=\frac{3\times 1760\times\cancel{36}}{\underset{100}{\cancel{60}\times\cancel{60}}}\times\frac{254}{100}\frac{\text{centimetres}}{\text{seconds}}$$

$$\fallingdotseq 134\text{ centimetres per second.}$$

(4) $$\frac{2\ \cancel{\text{yds.}}}{1\ \cancel{\text{sec.}}}=\frac{x\times 1760\ \cancel{\text{yds.}}}{60\times 60\ \cancel{\text{sec.}}}$$

$$\therefore\ x=\frac{\cancel{2}\times\overset{90}{\cancel{60}}\times\cancel{60}}{\underset{\underset{22}{\cancel{88}}}{\cancel{1760}}}=\frac{45}{11}=4\tfrac{1}{11}.$$

10. Acceleration.—The rate of change of velocity is called *acceleration*. Velocity may change in magnitude and

in direction, and the rate of either change is called acceleration.

When uniform, the acceleration in any direction is measured by the velocity gained in that direction in a certain time, divided by the time taken to gain it. When variable, its average value is measured in the same way.

Thus, if a falling body acquire a velocity of 96 feet per second in three seconds, its average acceleration is said to be $\frac{96}{3}$ or 32 *units of velocity per second.*

Hence acceleration bears the same relation to velocity as velocity did to distance; and denoting it by a, we have

$$a = \frac{v}{t}$$

as the algebraic statement of the measure of acceleration; remembering that v stands for the *velocity gained by the body in the time* t, and need not stand for any velocity actually possessed by the body. Thus the above falling body, instead of simply falling from rest, might have been thrown down from a balloon with an initial velocity of 100 feet a second; but if at the end of three seconds its velocity were 196, then its *gain* of velocity would be precisely the same as before, and its acceleration therefore still $\frac{96}{3}$ or 32. Hence, generally, v may be said to stand for the difference between the final and the initial velocities, which are conveniently denoted by v_1 and v_0 respectively, so that $v = v_1 - v_0$, and

$$a = \frac{v_1 - v_0}{t}.$$

11. When a body moving in a straight line acquires equal increments of velocity in equal intervals of time, its acceleration is said to be constant: for instance, a falling stone has constant acceleration; its velocity uniformly increases. It gains in fact a velocity 32 feet per second during every second of its motion. **In all that follows, the acceleration is supposed to be constant, unless it is otherwise stated.**

Note that acceleration is *velocity per time*, and that it is absurd to speak of an acceleration of so many *feet*, or even of an acceleration of so many *feet per second*, for this last is a velocity. An acceleration may properly be specified as so many *feet-per-second per second.* The expression for acceleration can be put in a fractional form; thus the acceleration

$$\frac{96 \text{ feet per sec.}}{3 \text{ sec.}} = \frac{96 \frac{\text{ft.}}{\text{sec.}}}{3 \text{ sec.}} = 32 \frac{\text{ft./sec.}}{\text{sec.}};$$ which last is commonly treated like an ordinary fraction, and briefly written $\frac{\text{ft.}}{(\text{sec.})^2}$; though it must be admitted that the notion of a squared second is absurd. There is, however, no principle involved in this mode of writing; it is merely an abbreviation, and it must always be interpreted as above.

The unit of acceleration appropriate to the C.G.S. system of units (see page 308) is $\frac{1 \text{ centim./sec.}}{1 \text{ sec.}}$ or $\frac{\text{cm.}}{(\text{sec.})^2}$; and the acceleration already expressed above in British units translates itself easily into C.G.S. units, by the knowledge that a foot equals 30·48 centimetres, thus:

$$32 \frac{\text{ft.}}{(\text{sec.})^2} = 32 \times \frac{30{\cdot}48 \text{ centim.}}{(\text{sec.})^2} = 975 \frac{\text{centim.}}{(\text{sec.})^2}.$$

Similarly, 981 centim.-second units equal 32·2 foot-second units very nearly.

12. If then the velocity of a body *increases*, its acceleration is the gain of velocity in each second of time; but if its velocity *decreases*, then the acceleration is really a retardation, and it must be reckoned negative, but as numerically equal to the loss of velocity in each second. Thus, suppose that in 3 seconds the velocity of a body changes from 196 to 100, its acceleration is -32. If the velocity of a body is constant, then of course its acceleration (or rate of change of velocity) is zero.

The Use of the Negative Sign.—It is a well-known method to

distinguish between opposite directions by opposite signs. Thus, if all distances measured to the right of any point be reckoned positive, any distance to the left will be negative, so that -30 feet will mean 30 feet to the left.

It is usual to reckon distances *up* as positive, and hence distances *down* as negative. The same may be extended to velocities, and a velocity upward may be called a positive velocity, a velocity downward a negative one. Thus, the velocity of a falling stone may be called negative, and it is continually getting *numerically* greater (though algebraically less): so the acceleration produced by gravity ought on the same convention to be called negative, because it is negative velocity which is added by it every second. In fact, an increasing negative quantity corresponds in algebra to a decreasing positive one, and *vice versâ*.

EXAMPLES—II.*

(1) A body starts from rest and acquires a velocity of 600 feet per second in half a minute. What is its acceleration?

(2) A body starts with a velocity 50 feet per second, and in $6\frac{1}{2}$ seconds has acquired the velocity 102 feet per second. What is its acceleration?

(3) A body moves with acceleration 32 ft./(sec.)^2, starting with a velocity of 20 feet per second. What is its velocity in 1, 2, 3, 6 seconds respectively?

(4) A train acquires a velocity of 20 miles per hour 5 minutes after leaving the station. What was its average acceleration during this time?

(5) How many miles an hour per hour is 32 feet a second per second?

(6) A train going 40 miles an hour is brought up in 40 seconds by the brakes. What is the rate of retardation in feet-per-second per second?

* All these are merely profit and loss questions. Velocity corresponds to capital, and acceleration to rate of gain. Thus Question 3 may be paraphrased thus: 'A man starts in business with £20, and gains £32 every year. How much has he got in 1, 2, 3, 6 years respectively?'

And No. 8 thus: 'A man starts with £128, and loses £32 annually. How soon will he have lost all? and what will he have in 1, 3, 5, 7 years?' Obviously he will have lost all in four years, and in seven years he will be £96 in debt.

A less simple kind of question is one that involves *distance*; for some examples, see Ex. IV. p. 29.

(7) A body starting with velocity 100 feet per second has only a velocity 52 in 4 seconds. What is its acceleration?

(8) A body with acceleration − 32 foot-second units starts with velocity 128. How soon is its velocity zero? and what is its velocity after 1, 3, 5, 7 seconds respectively?

(9) A body dropped from a stationary balloon falls with acceleration 32, and hits the ground with a velocity 512, the units being feet and seconds. How long was it in falling?

(10) The acceleration of a moving point expressed in terms of centimetres and seconds is 200. Explain exactly what this means. Find what number expresses the same acceleration in terms of metres and half-seconds.

(11) The acceleration of a falling body is 981 when referred to centimetres and seconds. If 1 foot were used as unit of length, and 1 minute as unit of time, how would this acceleration be represented?

1 foot = 30·48 centimetres.

(12) A train starts from rest, and 24 seconds later is moving at 240 yards per minute. Find its average acceleration.

(13) What acceleration is needed in order to get up a speed of 60 miles an hour in 2 minutes?

(14) What rate of retardation will destroy this motion in 5 seconds?

(b) *Curvilinear Motion of a Point.*

13. Besides change in the *magnitude* of velocity or rate of motion, there is another thing to be considered—namely, change in its *direction*. Hitherto we have only considered motion in a constant direction—that is, in a straight line; but when the direction of a point's motion is constantly changing, the path described is a curved line, or the motion is curvilinear. The rate of change of direction per unit length of a curve is called its **curvature**; and this again may be constant or variable. Most curves (the parabola, sect. 29, for instance) have variable curvature. A circle or helix has constant curvature. A straight line possesses zero curvature. The curvature of a circle is inversely proportional to its linear dimensions; because the angle which the

direction of motion turns through in going once round any circle is *four right angles*, which in circular measure is 2π (see sect. 14), and the curvature will be this angle divided by the distance travelled—that is, by the circumference, $2\pi r$; hence the curvature of a circle is numerically equal to the reciprocal of the radius, for

$$\frac{2\pi}{2\pi r} = \frac{1}{r} = \text{curvature of a circle.}$$

And the curvature at any point of any other curve is defined on the strength of this, as the reciprocal of the radius of that circle which coincides most closely with the curve at the point.

14. The *circular measure* of an angle at the centre of a circle is obtained by measuring or estimating the arc subtended by the angle and dividing this arc by the radius. The value of this ratio, *arc* ÷ *radius*, is independent of the size of the circle, and depends only on the angle to be measured. The angle subtended by an arc which equals the radius would, on this system of measurement, be denoted by 1, and is called a *radian*. It is about 57° 17′ 45″, being equal to $360° \div 2\pi$. The circular measure of any other angle is equal to the number of radians it contains. Four right angles, expressed in circular measure,

$$= \frac{\text{circumference}}{\text{radius}} = \frac{2\pi r}{r} = 2\pi,$$

where π denotes the ratio of the circumference of the circle to its diameter, and is approximately equal to $3\frac{1}{7}$, or, more nearly, 3·1416. Hence, 4 right angles $= 2\pi$ radians. We can now more fully state what is meant by curvature. For, as we have seen, the curvature of a circle equals

$$\frac{\text{angle turned through in going round the circle}}{\text{distance travelled}} = \frac{\text{4 right angles}}{\text{circumference}}$$

$$= \frac{2\pi \text{ radians}}{2\pi r} = \frac{1 \text{ radian}}{r}.$$

That is, the angle turned through by the direction of motion is 1 radian for each portion of the circumference travelled whose length equals the radius of the circle. For example, if the radius is 4 feet, a radian is turned through for each 4 feet of arc travelled, or $\frac{1}{4}$ radian for each foot, so that the curvature of this particular circle is $\frac{1}{4}$ radian per foot of arc travelled. The *numerical* value of the curvature is $\frac{1}{r}$, when the angles are measured in radians, but this expression must be considered as an abbreviation for 1 radian per arc-equal-to-radius, which finally in any given case becomes such and such a fraction of a radian per foot, or per inch, or per whatever unit of length is used.

15. A point moving in a curve, besides any acceleration it may have *along* the curve increasing its velocity, possesses an acceleration *at right angles* to the curve, or normal to the direction of its motion; this acceleration being proportional to the curvature of the curve, and affecting only the direction and not the magnitude of the velocity. Its magnitude is the rate at which velocity normal to the curve is gained by the point. This normal acceleration is called centripetal acceleration, and is further discussed in sects. 58–61, where it will be found to be proportional to the square of the velocity of the point as well as to the curvature of the curve; to be equal, in fact, to $v^2 \times \frac{1}{r}$.

Although the point is always *gaining* velocity normal to the curve or along its radius at this rate, it does not follow that it ever *possesses* any such velocity. It is in fact impossible for a point to possess any velocity except that along the curve, or at right angles to the radius of curvature; for as fast as velocity *along* the radius is generated, so fast does the direction of the radius change; in the same sort of way that a promise for to-morrow need never be fulfilled, because 'to-morrow never comes.'

II. MOTION OF AN EXTENDED BODY (ROTATION).

16. A point can only move along, it cannot spin; or rather, spinning makes no difference whatever to it or to its motion: but an extended body, whether it be a line, surface, or solid, may not only move bodily along or be translated; it may also turn round or rotate. The most general motion of an extended body is a combination of translation and rotation, but it is simpler to consider them separately. All that we have said about the motion of a point is equally true of the motion of an extended body so far as its *translation* is concerned; because, in simple translation, if we know the motion of any single point, we know that of the whole. Its rotation involves different ideas, which must now be considered briefly.

17. When a body rotates, every point of it describes a circle round some point or line which is the centre or axis of rotation.

The velocity of a point far from the axis is greater than that of a point nearer the axis; and in general every point has its own velocity, which is proportional to its distance from the axis, only points at the same distance having the same velocity; hence the 'velocity of a rotating body' is a meaningless expression. The number of times the body turns round in a second, however, is perfectly characteristic, and we must define some kind of *rotational* or *angular* velocity proportional to this.

To express the speed with which a body rotates, it is sufficient to specify the velocity of any one point together with its distance from the axis; for the velocity of the point, divided by its distance from the axis, is a constant quantity—that is, is the same for all points of the body at each instant, and is called the **angular velocity** of the rotating body. The velocity of every particle of the body is known in terms of this, for, being proportional to its distance from

the axis, it is equal to the 'angular velocity' multiplied by this distance; or, denoting the angular velocity by the letter ω, as is customary, the velocity of any particle at a distance r is

$$v = r\omega.$$

Angular velocity in rotations takes the place of ordinary velocity in translations. The name 'angular velocity' is given because it really represents the angle (expressed in circular measure, or radians) turned through per second by the whole body.

An example may render this more clear. The circle described by a particle at a distance r from the axis of a rotating body (say a nail on the circumference of a fly-wheel of r feet radius) is $2r$ feet in diameter, and hence $2\pi r$ feet in circumference. If the wheel turn round in T seconds, the velocity of the nail is $\frac{2\pi r}{T}$ feet per second; hence the *angular* velocity of the wheel is $\frac{2\pi}{T}$ radians per second, which is ω.

The most generally useful specification of angular velocity is in radians per second, as above, but it is usually first measured in revolutions per minute; and so a rapid mode of conversion from one measure to the other is often needed. Now, 1 revolution per minute equals $\frac{2\pi}{60}$ radians per second, and a good approximation to this number is $\frac{1}{10} + \frac{1}{200}$; which may be further improved by deducting $\frac{1}{4}$ per cent. from the result.

Thus, to convert 1264 revolutions a minute into radians per second, the work is as follows:

$$126{\cdot}4 + 6{\cdot}32 = 132{\cdot}72,$$

or, deducting $\frac{1}{4}$ per cent. of this, 132·39 radians per second. (A more exact value is 132·37.) To perform the inverse operation, multiply by 10, and subtract $4\frac{1}{2}$ per cent. of the result.

18. Of course angular velocity may be uniform or vari-

able; and if the latter, its rate of change, or increase per second, is called the **angular acceleration** of the body. Denoting this by α, we have $\alpha = \frac{\omega}{t}$ (just as we had $a = \frac{v}{t}$ in sect. 10). But $\omega = \frac{v}{r}$. Hence $\alpha = \frac{v}{rt} = \frac{v/t}{r} = \frac{a}{r}$; that is, the angular acceleration of a body is the acceleration of any particle divided by its distance from the axis. In other words, angular acceleration : acceleration : : angular velocity : velocity : : angle turned : distance travelled : : 1 : r.

EXAMPLES—III.

(1) What is the curvature of a circle 14⅔ yards in circumference?

It is numerically equal to the reciprocal of the radius in feet—that is, $\frac{2\pi}{44} = \frac{1}{7}$ of a radian per foot nearly; or your compass-bearings change about 8° for every foot you travel round such a circle.

(2) A point moves in the above circle with a constant velocity of 6 feet a second. What is its acceleration in magnitude and direction?

Its acceleration is always along that radius of the circle which passes through the moving point, and its magnitude is $\frac{36}{7}$.

(3) A point moving in a circle 8 feet in diameter has a velocity increasing by 18 every 3 seconds. What is the acceleration in magnitude and direction at different times?

There is a constant tangential acceleration equal to 6. The normal acceleration is zero at starting; at the end of the first second of motion it is $\frac{36}{4} = 9$; in two seconds it is 36; in three seconds, 81; and in t seconds it is $\frac{1}{4}(6t)^2 = 9t^2$. The actual acceleration at any instant is the square root of the sum of the squares of the tangential and normal accelerations at that instant; hence its direction, which at first is tangential, gradually swings round, so that in a few seconds it nearly coincides with the radius.

This explains what happens when we whirl a stone at the end of a string: it is necessary to start it with some purely tangential acceleration, obtained either by the help of gravity, or by a tangential push or pull. When once started, however, the speed may be increased to any extent by simply pulling the string a little to one side of the centre of the circle of motion, so that the tension in the string has both a tangential and radial component; and since the faster the stone is going, the smaller need the former be in comparison with the latter, it follows, that at a high speed the hand remains very nearly

steady in the centre of the circle; but it is really travelling round a small circle about a quadrant in advance of the stone—thus supplying the tangential force necessary to overcome the resistance of the air, even if the motion is not being accelerated.

Verify all this experimentally—whirling a weight in a horizontal circle on a flat table, in order to simplify matters by eliminating gravity.

(4) If the latitude—that is, the elevation of the N celestial pole—changes by 3 degrees while travelling a distance of 208 miles due north from Greenwich, what is the circumference of the earth?

(5) What is the angular acceleration of the moving point in No. 3?

(6) If a point describe a circle 5 feet in radius with an angular acceleration of 2 radians-per-second per second, what is its [linear] velocity at the end of 5 minutes from rest, and how many revolutions will it continue to make per minute if the acceleration then ceases?

(7) A wheel makes 20 revolutions per minute. What is its angular velocity in radians per second?

(8) A wheel possesses an angular velocity of 2 radians per second. How many revolutions per minute does it make?

(9) A point in the rim of a revolving wheel whose radius is 5 feet moves with a velocity of 6 feet per second. Find the angular velocity of the wheel, (1) in radians per second, (2) in revolutions per minute.

(10) A dogcart is travelling 16 miles an hour, and its wheels are 5 feet high. Find their angular velocity. Find also the speed with which the tire of the wheel is passing the elbow of the driver.

(11) A humming top, 6 inches in diameter, is started by a string of which 1 yard is wrapped round a spindle $\frac{1}{4}$-inch thick, and pulled off by a steady pull in 3 seconds. Find the initial angular velocity of the top, and the speed of its humming aperture.

(12) The wheel above the shaft of a coal-pit is 8 feet in diameter, and makes 90 revolutions per minute while letting down the cage. What is the speed of descent of the cage? Find also the angular speed of the drum in the engine-house, off which the rope is being unwound, if its diameter is 18 inches.

CHAPTER II.

CONTINUATION OF THE SUBJECT OF RECTILINEAR MOTION.

DISCUSSION OF THE STATEMENTS MADE IN CHAPTER I.

19. We have now obtained two definite statements, each of the nature of a definition—namely:

$$\text{Average velocity} = \frac{\text{distance travelled}}{\text{time taken in the journey}} \text{ or } V = \frac{s}{t}; \text{ and}$$

$$\text{average acceleration} = \frac{\text{velocity gained}}{\text{time taken in the acquisition}} \text{ or } a = \frac{v}{t}.$$

And we can proceed to reason on them, and trace their logical consequences, which will all be certainly consistent.

The treatment is very simple in the case of uniform acceleration. We desire to find the distance travelled during any interval of time when the initial velocity and the acceleration are given—that is, when the velocity at each moment is known. To calculate this distance s, we have to find the average velocity V, and then the first formula above gives us $s = Vt$. In the case of uniform acceleration, the average velocity is the arithmetic mean of the initial and final velocities. This can, perhaps, best be shown by an example. Thus, if the initial velocity had been 7 feet per second, and the final velocity, in t seconds, had been 19 feet per second, the average velocity according to the rule just stated would be $\frac{1}{2}(7+19)=13$ feet per second, and the distance travelled would be $13t$ feet. Now suppose the t seconds to be divided up into three equal intervals, the velocities at the beginning and end of each interval would be 7 and 11, 11 and 15, 15 and 19 respectively, since the velocity increases uniformly, and therefore, applying the rule, the average velocities during the intervals are $\frac{1}{2}(7+11)$, $\frac{1}{2}(11+15)$, $\frac{1}{2}(15+19)$—that is, 9, 13, and 17 feet per second respectively, and the duration of each interval is, by supposition, $\frac{1}{3}t$ seconds; $\therefore$ the distance travelled is $\frac{1}{3}t(9+13+17) = \frac{1}{3}t \times 39 = 13t$ feet as before. Hence the rule gives

us the same value of s at the end of the t seconds however we choose to divide it up. This would not have been true if the acceleration had not been uniform; as may be seen by repeating the process with different intermediate values, say for the set 7, 9, 13, 19. However, in all cases we shall have to deal with for the present, the velocity will increase regularly (i.e. the acceleration will be constant), consequently the average velocity is obtained at once by halving the sum of the initial and final velocities, $V=\frac{v_1+v_0}{2}$; using v_0 to stand for initial, and v_1 for final velocity (sect. 10).

Hence our first equation, which may be put into the form $s=Vt$, may be written more fully thus:

$$s=\frac{v_1+v_0}{2}t,$$

which signifies that in cases of uniform acceleration the distance travelled is equal to the half sum of initial and final velocities multiplied by the duration of the motion.

Similarly the second equation may be written in the form $v=at$, or more fully,

$$v_1-v_0=at,$$

which signifies that the excess of the final velocity over the initial velocity is equal to the gain per second multiplied by the duration of the constant rate of gain.

In case the final velocity is less than the initial, the gain becomes a loss, or v_1-v_0 is negative, and therefore also a is negative—that is, it is really a retardation, but it may still be called an acceleration, only a negative one.

20. Now let us study the two equations together, and see what we can get from them by any algebraical operation; remembering that algebra, like all other reasoning, never gives us anything absolutely and essentially fresh; it only brings out explicitly what is already contained implicitly in the *physical* statements which we subject to reasoning. The physical statements must be the results of the observation of nature, which is the only way of arriving at fundament-

ally new truths. Mathematical reasoning will, however, serve to bring out and make manifest what is really involved in the statements themselves when put together, if only we had sufficient insight to perceive it.

Our two statements or equations, written out fully, are

$$s = \frac{v_1 + v_0}{2}t, \quad \text{and} \quad a = \frac{v_1 - v_0}{t}.$$

First multiply the two left-hand members together and double the product, then do the same with the two right-hand members, then write the two products equal to each other (as of course they must be), and you get the new equation,

$$2as = (v_1 + v_0)(v_1 - v_0) = v_1^2 - v_0^2.$$

This is a relation between a, s, and v, without explicit reference to *time*, and it will often be useful.

Now try again, and this time get a statement not involving v_1, which we can do by substituting in the first equation the value of v_1 obtained from the second equation—namely, $v_1 = v_0 + at$,

and we get $$s = v_0t + \tfrac{1}{2}at^2.$$

Similarly we can get a relation excluding v_0, and it is

$$s = v_1t - \tfrac{1}{2}at^2.$$

21. But before proceeding to study the two equations together, we might have first made a simplification. An obvious simplification would occur if the initial velocity were made zero ($v_0 = 0$); in other words, if we agreed to consider only bodies starting from rest. In this case the gain of velocity v is equal to the final, v_1, and the average velocity V is equal to $\frac{1}{2}v_1$, which is now the same as $\frac{1}{2}v$; and so the two fundamental equations reduce to

$$s = \tfrac{1}{2}vt, \quad \text{and} \quad a = \frac{v}{t};$$

and the three derived from them simplify in like manner.

We thus obtain the following four equations between the distance travelled by a body *from rest*, the time taken in the journey, the acceleration, and the final velocity gained;

$$v = at.$$
$$s = \tfrac{1}{2}vt.$$
$$s = \tfrac{1}{2}at^2.$$
$$v^2 = 2as.$$

Of these, any two are independent statements, and the other two are logical consequences of them. The first of the four reads thus: The velocity gained in t seconds equals t times the velocity gained in each second. The second one thus: The distance travelled over in t seconds equals t times the average distance travelled over in one second (for this last is the meaning of average velocity). Both these statements are perfectly obvious. The other two statements cannot be put in quite so obvious a form. Observe that there are only four quantities involved, s, v, a, t, and that one of them is absent from each of the four equations.

22. The meaning of the second derived equation in sect. 20 is now clear. The space described by a body with the constant velocity v_0 is v_0t, and by one with the uniform acceleration a is $\frac{1}{2}at^2$; so the whole space described by the body possessing the initial velocity v_0, and *also* subject to the acceleration a, is

$$s = v_0t + \tfrac{1}{2}at^2.*$$

This may be regarded as a case of the composition of motions in the same direction. See sects. 24 and 73.

23. The results expressed by these equations may be made to appeal to the eye more directly, and thus be rendered easier to grasp, if illustrated by their analogy with geometrical diagrams.

If a horizontal line be considered as representing by its length a definite lapse of time—that is, if it be divided into a number of equal parts, each part representing say one

* If a and v_0 are of opposite sign, the subtraction is to be performed when the letters are arithmetically interpreted. The sign + means *algebraical* addition, which includes subtraction.

second; and if a vertical line represent by its length a certain velocity, by being divided into a number of equal parts, each part representing 1 foot-per-second; then if a body move with that velocity for that time, the distance travelled will be represented by the area of the rectangle contained by these two lines: that is to say, the number of feet travelled will be equal to the number of unit rectangles in the area, the height of each of which represents *a foot-per-second*, and the breadth of each *a second*. In a similar way a diagram can be made giving the velocity at each instant, and the distance travelled by a point moving in an accelerated or retarded manner.

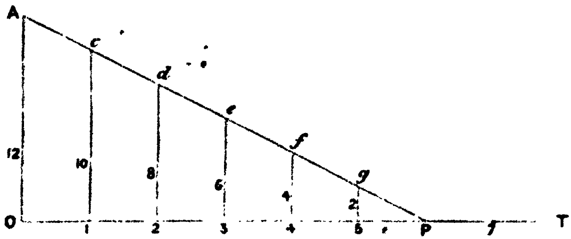


Fig. 1.

Thus in fig. 1, OT is the line of time, with the seconds marked off upon it. OA is a vertical line, and represents a velocity, say of 12 feet a second. If a body moved with this constant speed for 8 seconds, the distance travelled could be represented by the area of a rectangle constructed with base OT and height OA, because this area would be $12 \times 8 = 96$ appropriate units of area, and the distance travelled would be 96 feet.

Let a body start with this velocity 12, and lose 2 of it every second, then in 1 second its velocity will be represented by the line $c1$, in 2 seconds by the length of the line $d2$, and so on. Consequently, in 6 seconds the body will be at rest. The diagram thus represents, in a conventional and utterly non-pictorial fashion, a body starting with initial

velocity 12, and going with a uniform negative acceleration -2, till it stops. The average velocity would be 6, and would be represented by the length of the vertical line drawn in the middle of the time—namely, $e3$.

The distance traversed would be this average velocity multiplied by the time. That is, geometrically, $e3$ multiplied by OP, which is the area of the triangle OAP; for the area of a triangle is equal to the product of base and average height—in other words, to the product of half its height into its base.

Areas then in this figure represent distances. Or, more correctly, the *number of units of area* in one of these figures equals the *number of linear units* in the distance travelled. On this scale the area OAc1 represents the distance travelled in the first second; 1cd2, that travelled in the second second; 5gP, that travelled in the last second. The distance travelled in the three seconds between the first and fourth is represented by the area 1cf4, and so on—that is, each of these distances is equal to the number of units of area contained in the respective spaces. (Observe that the vertical scale and the horizontal scale in these, as in so many other diagrams, are quite independent of each other. The appropriate unit of area is therefore not necessarily a unit square, but a rectangle with unit sides.)

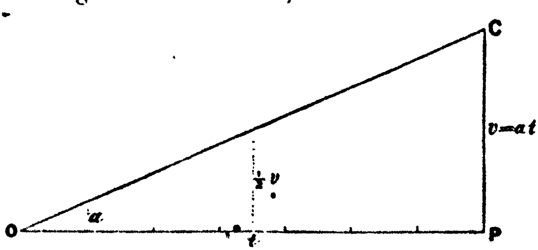


Fig. 2.

The representation of a body starting from rest with a *positive* acceleration is given in fig. 2.

The line of time is divided to represent seven seconds. The velocity gained in one second is represented on some convenient scale by the line marked *a*, which therefore represents the numerical value of the acceleration. The velocity gained in the whole time is marked *v*; it is obviously equal to $7a$ or at. The dotted line in the middle of the time is the average velocity, and it is evidently $\frac{1}{2}v$.

The area of the whole triangle represents the whole distance travelled, and it is half the height multiplied by the base, or $\frac{1}{2}\ v\ .\ t$, or, what is the same thing, $\frac{1}{2}\ at\ .\ t$, that is, $\frac{1}{2}\ at^2$.

The little left-hand triangle is numerically equal to $\frac{1}{2}\ a$ in area (its base being unity), and it represents the distance travelled in the first second.

The velocity possessed by the body at any second or fraction of a second is found at once, simply by measuring, and interpreting on the proper scale, the vertical height of the triangle at the place defined by the time. The whole problem is in fact geometrically represented.

If the body started with an initial velocity, and then went on with increasing velocity, its motion would be represented by fig. 3, which is supposed to represent what happens in three seconds. The initial velocity is marked v_0, and the final v_1; the latter being made up of two parts, the gain of velocity at, and the original velocity v_0. The rest is marked as before, the base represents t, and the whole area represents the whole distance, s, travelled in the three seconds—namely, v_0t, the rectangle, *plus* $\frac{1}{2}\ at^2$, the triangle on the top of it.

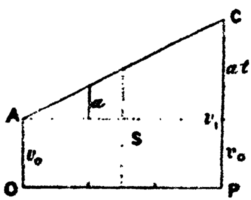


Fig. 3.

The dotted line in the middle is of height $v_0 + \frac{1}{2}\ at$, or, what is the same thing, $v_1 - \frac{1}{2}\ at$; and therefore it is

$\frac{1}{2}(v_0 + v_1)$, or the average velocity. Any one of these three expressions for average velocity when multiplied by the time will give the distance travelled (cf. equations of sect. 20). If the initial velocity be negative, the line representing it must be drawn *down* from the line of time, instead of up.

These diagrams will be found exceedingly useful and conducive to clear ideas, as soon as a little practice has made you familiar with them. For some more illustrations of their use, see sect. 69, which can be read now; for the commonest example of uniform acceleration is that caused by the earth's attraction, which causes all freely falling bodies to acquire a speed of 32 feet a second in every second of their fall—in other words, which causes a uniform acceleration of 32 feet a second per second.

EXAMPLES IV.*

(1) In Examples II., 1–9, find the distance travelled by the different things in the times given.

(2) A body starting from rest, and travelling 63 feet in a straight line, gains a velocity of 81 feet per second. What is its acceleration? (*The most direct way to get the Answer is to use the formula* $v^2 = 2as$.)

(3) What is the acceleration of a body whose velocity changes from 7 to 21 while it travels 100 feet? (*Use the formula* $v_1^2 - v_0^2 = 2as$.)

N.B.—The arithmetic is often simplified by taking the difference of two squares in the form of the product of sum and difference.

Questions (2) and (3) may also be solved by dividing the velocity gained by the time taken in gaining it; the time being found by dividing the distance travelled by the average velocity.

(4) Find the accelerations of the following bodies:

A,	whose velocity changes from	15 to 5	in going	50 ft.
B,	" "	15 to −5	"	50 ft.
C,	" "	−5 to 15	"	50 ft.
D,	" "	120 to 0	"	640 ft.

(*Remember that the square of a negative number is positive.*)

The extreme distance from the starting-point attained by B is 56¼ feet, but 6¼ feet of this is retraced. It therefore takes longer in the journey than A did, but its acceleration happens to be the same.

Similarly with C, the first thing it does is to go $6\frac{1}{4}$ feet backwards and come to rest for an instant; then it retraces its path and goes 50 feet forwards, where the question leaves it; but it is still going on with a speed increasing by 2 in every second.

(5) Find the time of the motion in all these cases, and draw a diagram for the several motions.

Begin by drawing the line of time; then draw verticals for the initial and final velocities, paying attention to sign, and join the extremities of these lines; then study every part of the diagram, and note its connection with the equations.

(6) A train with the brakes on, moving with acceleration -3, has a velocity 78 when passing a particular station. How much farther will it go?

(7) A point moves 16 feet in 1 second and 20 feet in the next. How long has it been moving with uniform acceleration since it started from rest, and what is the rate of the acceleration? Also, how far would it go in the next 12 seconds of its motion, and when will its velocity be 128?

(8) A body slackens speed from 50 to 30 feet a second in going 20 yards. Find how soon it will stop if the same rate of retardation continues.

(9) The velocity of a train, moving with uniform acceleration, is, at three points A, B, and C, 40, 50, and 60 miles an hour respectively. The distance AB is 27 miles. Find the distance BC.

(10) A train going 40 miles an hour is brought up in 200 yards by the brakes to avoid a collision. What is the acceleration in miles-per-hour per second, and also in feet-per-second per second?

(11) What acceleration is needed to get up a speed of 60 miles an hour in a half-mile run? What is the brake retardation that could destroy this motion in a train-length, say 100 yards? What retardation would a stoppage in 5 yards represent?

The following examples are cases of uniform acceleration under gravity. The acceleration may be taken as 32 feet-per-second per second. The resistance of the air is neglected.

(12) A body falls from rest. Find the distance travelled in 5 seconds, and how far it will go in the next second.

(13) A falling body describes 100 feet in the last second of its motion. Find how far it must have fallen, and also the time taken.

(14) A bullet is dropped at a place where the intensity of gravity is only 20. How many feet will it have fallen in 4 seconds; and how far will it go in the next second?

(15) Express by means of a diagram, connecting velocity and time, the motion of a body which is thrown up into the air, rises 256 feet, and falls to the earth again.

(16) A stone is thrown vertically upward with a velocity of 40 feet a second. How high will it rise? and how long will it be before it returns to your hand? If you let another stone drop down a well, at the instant the first is within 20 feet of your hand on its return journey, at what distance below your hand will the two bodies meet?

(17) A bullet is dropped from the top of a tower 100 feet high, and at the same instant another bullet is thrown vertically from the bottom of the tower with velocity just sufficient to carry it to the top. Show where and when the bullets will pass each other.

(18) If you throw up a cricket ball and catch it after 5 seconds, how high will it go, and with what velocity will it return to you?

Composition of Motions in General.

24. When a body has several motions given to it at the same time, its actual motion is a compromise between them, and the motions are said to be compounded, the actual path taken being called the resultant. Thus, suppose a fly to crawl along a tea-tray from A to B (fig. 4), while at the same time some one pushes the tray along a table a distance PQ; the fly will then have two motions, and its actual motion with reference to the table is the resultant of the two motions, AB and PQ. To find where the fly is at the end of the two motions, we must observe where

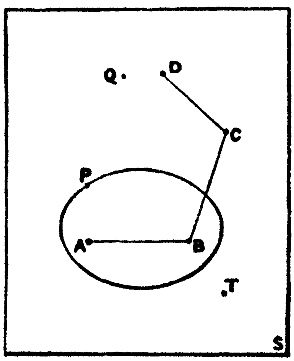


Fig. 4.

the point B of the tray has gone to, for the fly has crawled to B; but B has been moved to a point C, such that $BC = PQ$. Hence the fly is at C, and its actual motion must have been along some path AC

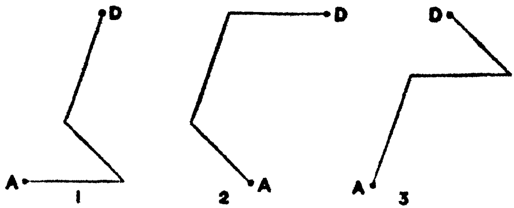


Fig. 5.

(not necessarily a straight line); and AC is therefore called the resultant of the two motions, AB and BC. If, besides these two, the table itself had been pushed in the direction ST, or what is the same thing, CD, then we should have had three motions to compound; and, as the fly would have got ultimately to D, AD would have been the resultant of the three motions. The *order* in which the steps are added evidently does not matter, for the same point D is arrived at by taking the table motion before that of the tray, as in 1, fig. 5; or the fly's proper motion after both the others, as in 2; or the fly's motion between the other two; or in any other of the six possible orders in which three motions can be compounded.

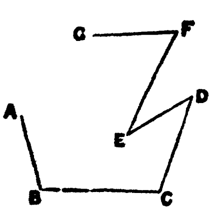


Fig. 6.

And so we readily see the rule for compounding any number of motions. Draw lines, or cut pieces of stick,* representing each motion in magnitude, direction, and sense,† and lay these

* See footnote to page 135.

† That is, make some difference between the two ends of the line, indicating by an arrow-head or otherwise which way the motion takes place in the given direction.

lines or sticks in any order, with the end of one coinciding with the beginning of the next (the lines may be moved into any positions, provided each is kept parallel to itself); then some line joining the first point of the first with the last point of the last, must be the resultant of the whole set of motions.

Thus, some line AG is the resultant of the six motions, AB, BC, CD, DE, EF, FG. This proposition is called the polygon of motions, because the resultant is represented by the line required to complete a polygon.

As a matter of fact, however, the sides of the polygon need not necessarily be straight lines. The end points of the line are the only essential matter when one is dealing with simple change of position without regard to time or speed.

25. The composition of *two* motions, AB, BC, into a third, AC, requires only a three-sided polygon, so it is often called the **triangle** of motions. Or if we choose to represent the two component motions, AB, BC, by lines, AB, AB′, drawn from the same point, we get the **parallelogram** of motions, which is merely a less simple, but sometimes convenient, way of regarding the triangle of motions. The resultant motion is the diagonal of a parallelogram whose two adjacent sides represent the component motions.

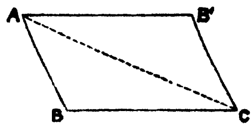


Fig. 7.

26. This law, by which two motions are compounded, is of very frequent occurrence in all parts of mechanics, and is referred to as **the parallelogram law**. It may be stated thus: If two causes act on a body at once, or if a body experience two simultaneous effects in different directions, then if these effects are represented in magnitude and direction by two adjacent sides of a parallelogram, the effect experienced by the body is called the *resultant* effect, and is represented, on the same scale, by the concurrent diagonal

of the parallelogram—that is, the diagonal which passes through the point of intersection of the two sides; or, it is the same effect as would be produced by a *resultant cause* represented in magnitude and direction by the similarly situated diagonal of a parallelogram whose two sides represent the component causes; provided always that the causes or the effects can be shown to be of such a nature that this law is applicable to them.

Composition of Uniform Velocities.

27. So far we have only studied the composition of changes of position; now let us study the composition of velocities—first, when uniform. Remember that a velocity is numerically equal to the space described in 1 second.

Let a body start from O (fig. 8) with two velocities, one horizontal and of magnitude Oa, the other vertical and of magnitude Ob. Then Oa and Ob represent the distances travelled in 1 second in the respective directions, and consequently at the end of 1 second the body is at the point c, the opposite corner of a parallelogram with sides Oa and Ob; hence the body must really have travelled the distance Oc in 1 second, therefore Oc is its *resultant velocity* in magnitude and direction. In 2 seconds it will have travelled horizontally to a', and vertically to b', and therefore it will really have reached c'. And it is easy to see, by drawing or otherwise, that the straight line Oc' passes through the point c, and that Oc' equals twice Oc (because Oa' = twice Oa, and Ob' = twice Ob); or the distance travelled in 2 seconds is twice the distance travelled in 1; and so, generally, the resultant of two uniform velocities is another uniform velocity along the diagonal of the parallelogram whose adjacent sides represent the components.

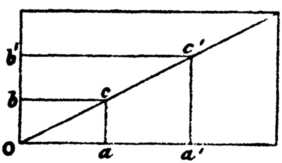


Fig. 8.

Hence the resultant of two velocities is obtained by precisely the same parallelogram law as the resultant of two simple motions. Similarly the 'polygon' law is applicable for compounding any number of velocities greater than two.

Composition of Uniform Accelerations.

28. Accelerations may evidently be compounded by the same law as velocities, because acceleration is the velocity gained per second. Thus let a body be subject to any two accelerations, say a horizontal one Oa, and a vertical one Ob (fig. 8); then Oa and Ob represent the velocities gained per second in these two directions respectively, and therefore the actual velocity gained in the time is Oc; in other words, Oc, the diagonal of the parallelogram, measures the resultant acceleration. Hence accelerations are compounded by the same* law as velocities.

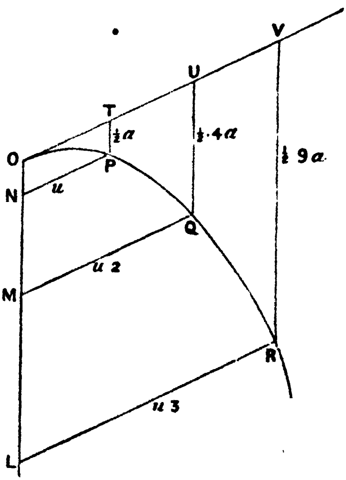


Fig. 9.

29. *Combination of a uniform Velocity with a Velocity uniformly accelerated in a constant direction.*—Let a body start from O with a uniform velocity u in some direction OV (fig. 9), and a uniform acceleration a in some other direction, such as OL vertically downwards; then in successive seconds the distances traversed in the first direction will be numerically equal to

$$u, \qquad 2u, \qquad 3u, \qquad 4u, \qquad \&c.;$$

so that, if this constant velocity u were the only one

possessed by the body, the body would be at T after 1 second, at U after 2, at V after 3, and so on (fig. 9.) But the uniform acceleration is acting at the same time, and *causing the body to descend a height proportional to the square of the time* ($\frac{1}{2}at^2$); *hence in successive seconds the vertical distances traversed will be*

$$\frac{a}{2},\ 4\frac{a}{2},\ 9\frac{a}{2},\ 16\frac{a}{2},\ \&c.,$$

bringing the body to the level N in 1 second, M in 2, L in 3, and so on, if it had acted alone.

The *actual* position of the body, therefore, at the end of successive seconds will be found by completing the parallelograms,

OTPN,
OUQM,
OVRL, &c.;

the result being that the body reaches the point P in 1 second, Q in 2, R in 3, and so on.

Now the simplest continuous curve which can be drawn through these points, OPQR, &c., is a parabola, and this is the actual path of the body.

It will be shown later (sect. 74) that this is the path of a projectile thrown in *vacuo* in the given direction with the velocity u, and subject to gravity, which causes a uniform downward acceleration. It is well seen in the curve of a steady jet of water, for each drop of water takes this path. It is also illustrated by Morin's machine (sect. 75).

This example illustrates the fact, noted at the end of sect. 23, that the resultant motion (the diagonal of the parallelogram) need not be represented by a straight line. It will be straight if the two things compounded are both uniform; otherwise it will in general be curved..

EXAMPLES—V.

(1) A point has two motions, one east with a uniform velocity 30, the other north with a uniform velocity 40. What is its actual motion?

(2) A boat is rowed at right angles to the banks of a straight river, at a pace half as fast again as the stream flows: it reaches the opposite bank 2 miles below the starting-point. Find the breadth of the river and the distance rowed.

(3) *A point describes a circle with a constant velocity v, and at the same time the centre of the circle moves forward in a straight line with the same velocity.* What is the motion of the point?

N.B.—This is the case of a nail on the circumference of a coach-wheel. The point describes a curve with cusps, called a cycloid; its velocity when at the top of the wheel is $2v$, and when on the ground is zero; its velocity at the extreme right and left points of the wheel is $v\sqrt{2}$; its velocity is v at two points whose distance from the ground is half the radius.

Resolution of Motions.

30. The inverse process to that of 'compounding' is called resolving, and is an operation which, in practice, is found extremely useful. We have seen that a pair of motions (or velocities or accelerations), one in a vertical and the other in a horizontal direction, compound into a single motion in a slant direction, wherefore it follows that, when we choose, we may analyse or resolve a slant motion (or velocity or acceleration) into a pair of components, one of which may be horizontal while the other is vertical, or both of which may have any definite directions we please provided we affix to each its appropriate magnitude. All that is essential is that the two components shall be represented by the sides of a parallelogram, of which the diagonal represents the thing whose resolution is sought. We shall study this process in greater generality hereafter (see sect. 112 and fig. 25); for the present, it will be best to illustrate the use of resolution by an example.

Suppose a body thrown in a direction slanting upwards at an angle of 45° with a velocity of, say, 141·42 feet a second. This velocity may be considered as the diagonal of a square whose sides each represent 100 feet per second, and the whole motion may be considered as analysed into

two parts—a horizontal part, which continues uniform but for friction; *and a vertical part, which is constantly subject to the downward accelerating force of gravity, which destroys its upward velocity, and generates downward velocity at* the rate of 32 feet a second every second. Hence such a projectile will ascend in a curved path for $\frac{100}{32}$ or about 3 seconds, attaining a height of about 150 feet and travelling horizontally about 300 feet in the same time; then its curve will begin to slant down, and ultimately it will arrive at its original level, about 600 feet from its starting-point, after the lapse of about 6 seconds. If the ball were shot in any other direction, its motion could be treated and its velocity resolved in like manner, but its initial vertical and horizontal components would no longer be equal. We may return to this subject in sect. 76.

Relative Motions.

31. So far the pair of motions compounded or combined have really belonged to the body under consideration, but there are cases where a practical problem is simplified by the device of attributing to a body motion which it does not really possess. The most frequent example of this device is the assumption that the earth is at rest; for this is attributing to it a motion which it does not really possess (for rest is of course a particular case of motion), and problems are certainly simplified thereby. For instance, when we say, as above, that the path of a projectile is a parabola, we are ignoring the motion of the earth; we are not thinking of what is called the *absolute* motion of the projectile—that is, its velocity through space or through the ether; we are thinking of its velocity relative to the earth. Our ignorance of absolute velocities makes this notion of relative velocity not only convenient but essential; and

being thus compelled to employ the notion, we may proceed to make further use of it.

Now, in treating of the simultaneous motion of two bodies, say of two ships at sea or of two impinging balls, it may be sometimes convenient to ignore their motions with respect to water or ground, and attend only to their motion relative to each other. This relative motion can be brought out and displayed most clearly by the artificial device of imagining both bodies to be affected with a fictitious velocity which is to be compounded with their real velocities; for a fictitious velocity may be chosen so as to be equal and opposite to the real velocity of one of the bodies, and in that case the resultant velocity of that one becomes zero, while the resultant velocity of the other body becomes its relative velocity with respect to the one thus imagined to be stationary.

There is no important principle underlying all this; it is a mere practical device for simplifying problems in relative motion which otherwise would be more complicated for clear mental grasp.

As an illustration, consider two ships sailing, one (B) due north with velocity v, and the other (A) due east with velocity u; let us observe them in some definite position, say at distances b and a respectively from the point O at which their paths will hereafter cross (fig. 10), and let us ask when and where they are in most danger of collision—that is, when and where and what is their shortest distance. If the velocities are in simple proportion to the distances from the crossing-point—that is, if $u:v=a:b$, they will certainly collide, unless one of them alters her course; but if the speeds are in some other proportion, they will pass within a certain minimum hailing distance. To find this distance, the neatest way is to suppose a fictitious velocity, $-v$, impressed upon the whole system; by this means B is brought to imaginary rest, and A is sent moving (imaginarily) with velocity $\sqrt{(u^2+v^2)}$ in another

track indicated in the figure by the line AP; and a perpendicular on to this line from B—namely, BP—represents the nearest approach of A to B. This is the shortest distance required, but it is not in the true locality. The real condition of the problem limits B to the north course and A to the east course, hence their real positions when hailing are B′ and A′ respectively, where A′B′ is a line drawn equal and parallel to PB.

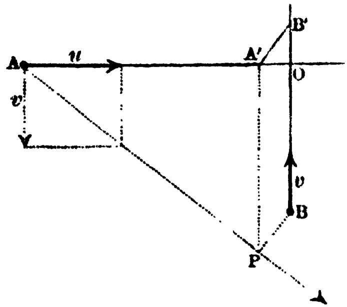


Fig. 10.

Thus the problem is solved in all essential features, even for those who do not care to take the trouble to work out the arithmetical details. By drawing the above figure to scale, the solution can be simply constructed; whereas, if both bodies were contemplated as moving in their true paths, a direct geometrical construction for the result would be hardly possible, and arithmetic would be necessary.

Applying kinematical considerations to the above figure, and denoting AA′, BB′ by x, y respectively, we see that if a time t elapses between the initial positions A, B, and the desired positions A′, B′,

$$x = ut, \tag{1}$$

$$y = vt; \tag{2}$$

and the distances of A′ and B′ from O, the crossing-point of the tracks, are, in the figure, $a - x$ and $y - b$ respectively. (If A′ had been beyond O, and B′ not up to O, the distances would have been $x - a$, and $b - y$ respectively.)

We may further see that the triangle A′OB′ is similar to the triangle PA′A, because the corresponding sides are perpendicular; therefore $(a - x) : (y - b) = y : x$. (3)

These three equations express all the essential kinematic and geometrical facts; and simple algebra will now extract the information required—namely, the place, time, and distance corresponding to the positions A′ and B′. The solutions are:

$$\text{shortest distance } A'B' = \frac{av - bu}{\sqrt{(u^2+v^2)}} = s, \text{ say};$$

$$a - x = s\frac{v}{\sqrt{(u^2+v^2)}} = \frac{v}{u^2+v^2}(av - bu);$$

$$y - b = s\frac{u}{\sqrt{(u^2+v^2)}} = \frac{u}{u^2+v^2}(av - bu);$$

$$t = \frac{au + bv}{u^2+v^2}.$$

EXAMPLES—VI.

(1) The courses of two steamers are at right angles to each other, and their speeds are 12 and 16 miles an hour respectively. If they are at first, both of them, 1 mile distant from the point where their tracks will cross, find how near they will approach each other and how soon.

(2) A lady wishes to cross a muddy road, 30 yards wide, by the shortest path, at 3 miles an hour, in front of a vehicle which is coming along the middle of the road downhill at 9 miles an hour, but which has still a distance of 50 yards to go before reaching her level. By how much will she escape the vehicle? Also, what is the least speed at which she need walk if she ignores the mud and takes a slanting direction?

(3) Two trains are moving, with velocities of 25 and 40 miles an hour, along two lines inclined at an angle of 60°, and are respectively 300 yards and 200 yards from the crossing-point. Represent completely the motion of either train as it appears to the passengers in the other.

(4) Find by a graphical construction drawn to scale, and also by calculation, the resultant of the following velocities which are communicated to a point—namely, 10 feet per second in an easterly, 20 feet per second in a north-easterly, and 30 feet per second in a northerly direction respectively.

(5) If a point has a velocity of 1 foot per second to the east, and also a velocity of $\sqrt{3}$ feet per second to the north, determine the velocity which must be compounded with these to bring it to rest.

(6) A steamer, travelling at 20 miles an hour, has to travel north-east through a current flowing south at 5 miles an hour: show, in a diagram, the direction in which it should be steered to allow for the current, and measure, or calculate, how far it will have travelled from its starting-point in 3 hours.

(7) If the steamer, through ignorance of the current, is steered north-east, find how far it will be carried out of its course in 3 hours, and how far it will be from its starting-point.

CHAPTER III.

ON QUANTITY OF MATTER AND QUANTITY OF MOTION.

A CHAPTER OF DEFINITIONS.

(I.) MOTION OF A PARTICLE, OR TRANSLATION.

(Inertia and Momentum.)

32. We have so far studied motion in the abstract, with reference to its direction and its rate, but without reference to the body moving, or to the amount of motion possessed by it. Let us now consider what is meant by this last phrase 'amount' or 'quantity of motion.'

First, it is plain that in any actual case of motion there must be some matter moving; and it will be sensible and consistent with the ordinary use of language to consider the quantity of motion in a body as proportional, first, to its speed, and, secondly, to its quantity of matter; and this is the scientific custom.

33. Now we understand what is meant by speed, but what do we mean by quantity of matter? First of all, of course, a large solid ball contains more matter than a small one of the same material; but quantity of matter does not depend on size alone, it depends also on the closeness or density of the substance. A small iron ball may contain more matter than a large cork one.

Now matter possesses a certain characteristic property called 'inertia,' or power of reacting against a force applied to change its state of motion. It is on account of this property that *force* is required to move matter or to check its motion—the passive resistance or reaction of the matter

itself against whatever is forcing it to move being called its **inertia-reaction** or **inertia-force.** Thus a railway truck has great inertia, because considerable exertion is required to stop it or to set it going, even on a level line, with friction reduced to an insignificant item.

This fact of 'inertia' was expressed by Newton in the following 'law' or axiomatic statement: '*Every body perseveres in its state of rest or of moving uniformly in a straight line, except in so far as it is made to change that state by external forces.*' This is often referred to as Newton's FIRST LAW OF MOTION, or as *the Law of Inertia*; and it is equivalent to defining force as that which causes change of motion in matter. Its essence can be briefly expressed by saying that *without force there can be no change of motion*; or *the motion of a body is constant both in magnitude and direction except when a force is acting on it*; or, again, that *when resultant force is zero, acceleration is also zero.* That a body should thus persist in its state of motion (of which rest is only a particular case) unless some cause acts to change its state, is in accordance with the fundamental or common-sense axiom that no effect happens without a cause. It is not likely that a piece of matter should itself be able to change *its own* state, whether of rest or motion: some external cause or influence from other bodies is necessary. The above law asserts that the sole cause of change of motion in matter is *force*. It may be taken as a definition of force in terms of matter, or a definition of matter in terms of force. Since we have a direct sense of force (the muscular sense), as stated in the Introduction, the latter is the most useful definition: hence matter is defined as that which requires muscular action or its equivalent to change its motion; in other words, matter is that which possesses inertia. This is the sole Newtonian test of matter; any reference to gravitation, attraction, or weight is beside the mark. If it be asked whether elec-

tricity or ether is matter, we must inquire whether either of these entities requires force to change its state of motion. The answer may or may not be an easy one to obtain by experiment, but there is no haze about the question. If the ether possesses inertia it is a form of matter, if it does not possess inertia it is something else. Heat possesses no inertia; it is propelled by difference of temperature, not by mechanical force, hence heat is not a form of matter; and so on.

34. Since inertia then is a characteristic property of all matter, it will serve to *measure* the quantity of matter in any given mass, and it is always used for this purpose in Dynamics. Suppose you have a number of smooth cubes or blocks, each made of a different material but of the same size, resting on a perfectly smooth horizontal table, and you give them each a little push of exactly the same strength; the push will have the least effect on those which contain the greatest quantity of matter. Thus imagine four of the cubes to be of cork, wood, iron, and gold respectively, and that you give each a sudden knock. The cork block would be considerably affected, and would slide off the table; the block of wood would be affected next in extent; while the iron and gold blocks would perhaps hardly be stirred, but whatever movement there were would be greater in the iron than in the gold. We should hence conclude that the gold block contained most matter, the iron next, and the cork least. This is a perfectly direct and scientific method of comparing the masses of bodies, and more than *comparing*, for it is capable of affording a definite *measure* of the quantity of matter in a body. Thus either apply the same force for the same time to each body, and measure the velocity imparted (if the same velocity is imparted to a number of bodies by the same shock or impulse, they have all the same inertia, and therefore the same quantity of matter); or graduate the forces applied to the different bodies, so that

each may move with the same acceleration, the forces required will measure the inertia of the several bodies. The forces themselves must be measured by the strain method (see Introduction, sect. 6), as the other method would lead to reasoning in a circle.

Fig. 11 shows the experiment carried out as far as it is possible to carry it out without a perfectly smooth table. The blocks are mounted on rollers to diminish friction, and are attached each to a stiff spring balance, such as some of those made in dial form by Salter, which will yield in a very small but yet a measurable degree, since the yield is magnified by the index. The simple form shown in the figure yields too much to be suitable. These balances are then all tied to a rod, and are pulled quickly along, so that all the blocks have practically the same acceleration imparted to them. The springs indicate by their stretch the inertia-reaction of each body.

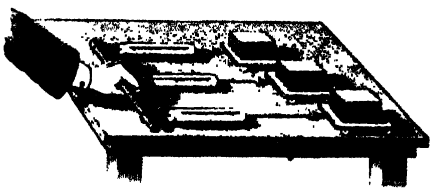


Fig. 11.

35. One often actually applies this method of comparing masses in common life. Suppose you see a cask lying on level ground, and wish to know whether it is full or empty; you give it a kick or a push with your foot, and if it yields and moves easily, you conclude that it contains very little matter—that is, that it is empty; whereas if it almost refuses to move, it must contain much matter; and if it contains *dense* matter, such as iron or lead, it will be harder to move than if it contained, say, earthenware, and this again harder than if it were full of straw. Hence we find that the *quantity of matter* in a given body, as measured by its inertia, depends first on the *density* of its material; and secondly, on its size or *volume*. And we might define quantity of matter as the product of volume and density, giving this product the name of *mass*. The 'mass' of a body hereafter, then, shall stand for the quantity of matter in it, and shall equal its volume

multiplied by its density. This last serves strictly for a definition of *density* rather than of mass, as thus:

$$\text{Density} = \frac{\text{quantity of matter in body}}{\text{volume of body}};$$

or, more shortly, density = mass per unit volume. The simplest unit of volume is the cube of the unit of length—say a cubic foot or a cubic centimetre; and density will be expressed as so many pounds per cubic foot, $\frac{\text{pounds}}{(\text{foot})^3}$, or so many grammes per cubic centimetre. The numerical value of the density of water on the first system of units is about 62; on the second system it is about 1.

We see, then, that mass is measured, and must be held to be defined, by the property of inertness possessed by matter—that is, by its requiring force to move it if at rest, and to stop it if in motion. This idea of the muscular effort needed to set a body moving or to stop it, must be held to be the primitive idea of inertia. The greater the effort required to produce a given motion, the greater the inertia; and as every particle of matter possesses this property, the more particles there are the greater is the inertia, and inertia is the only *direct* measure of mass in mechanics.

To recapitulate, then, mass means quantity of matter, and is measured by inertia.

36. Just as the standard of length is an arbitrary distance, called in this country a standard yard, and defined as the distance between two marks on a certain bar of metal at 62° F., so the standard of mass must be an arbitrary quantity of matter. In this country the standard mass is one of several equal masses of platinum kept in the Houses of Parliament, the Mint, and other places, from which copies are taken for general use, and it is called a *pound* avoirdupois.

The metric standard of mass is a similar mass of platinum kept at Paris, and called a kilogramme. A kilogramme is

about 2·2 pounds. The thousandth part of this (a gramme) is commonly used as the practical unit of mass for scientific work all over the world, and the system of units based on the centimetre, the gramme, and the second is called the C.G.S. system. Similarly, the common British system of units, based on the foot, the pound, and the second, may be called the F.P.S. system, and units in that system may be called F.P.S. units.

The student must avoid confusing the mass of a body with its heaviness or weight. A pound is the British unit of mass, but because the pound happens to pull downwards with a certain force (*avoir*, in fact, *du pois*), people often think of this *pull*, *force*, or *weight* as the essential thing, whereas it is quite a secondary thing. When we speak of this force, we shall call it the pound-*weight*, or the weight of a pound: it is not the pound itself (see sect. 64). Suppose you wish to leave some flowers to be pressed all night in a book, and you put on the book for the purpose a few pound or other weights; what you are then concerned with is the *weight* of the pounds, or their pull downwards. But suppose you buy six pounds of sugar or of soap; what you are *then* concerned with is the quantity of matter or *mass* which you obtain, and the force with which the matter tends downward is a secondary, and sometimes a burdensome consideration. This confusion has arisen from the fact that the shopman measures the mass out to you, not by a direct method like that shown in fig. 11, but by an indirect, though practically simpler method, founded on the attraction of gravitation, which is proportional to the masses of the attracted bodies within the limits of experimental error. So that the shopman compares not the *mass* but the *weight* of your purchase with his standards. We must try, however, to avoid confusing *mass* with *weight*, even at the risk of a little pedantry, which may be necessary until we are quite clear on the subject.

37. Now we have already seen (sect. 32) that it is reasonable to define *quantity of motion* as directly proportional to the quantity of matter (or mass) moving, and to its rate of motion (or velocity). Hence let us at once define quantity of motion as equal to the *product* of the mass in motion and its velocity. The name given to quan-

tity of motion is *momentum;* so we have now the definition:

$$Momentum = mass \times velocity, \text{ or } M = mv,$$

where m stands for mass, and M for momentum.

Momentum *means* quantity of motion, and *is measured by* the quantity of moving matter multiplied by its velocity.

38. The unit of mass being a pound, the unit of momentum must be that quantity of motion possessed by a pound of matter when moving with a velocity of one foot per second. The momentum of a $\frac{1}{4}$-lb. cricket-ball moving at the rate of 56 feet a second, is $\frac{1}{4} \times 56 = 14$—that is, fourteen British units of momentum, as just defined. The C.G.S. unit of momentum is that of 1 gramme moving at the rate of 1 centimetre per second.

Thus, the jet of a fire-engine which is delivering a cubic metre of water (1000 kilogrammes) every minute, at a speed of 6 metres per second, possesses momentum to the amount of ten million C.G.S. units in each six-metres length of it, and transfers this amount of momentum per second to any fixed obstacle against which it impinges. The result, we shall hereafter learn (sect. 48), is a steady pressure of ten megadynes, a little more than ten kilogrammes weight.

The momentum of a 50-lb. cannon-ball moving with a velocity of 1612 feet per second, is 80,600 foot-pound-second (F.P.S.) units.

That of a three-ton truck (1 ton = 2240 lb.), moving at the rate of 12 feet per second (roughly about eight miles an hour, see Ex. I. 4), would be 80,640 F.P.S. units, or nearly the same as that of the above cannon-ball.

Now we shall find in the next chapter that an impulse or propulsion is proportional to the quantity of motion it causes; hence we see that in some sense or other the same motive power was required to set the above cannon-ball going as was required to set the truck, for both possess the same quantity of motion. Yet the force exerted by the powder in the cannon was undoubtedly greater *while it lasted* than the force exerted by the horse or engine, or whatever started the truck; but then the former acted for the fraction of a second only, while the latter took perhaps a minute. What is called the *impulse* or propulsion of the force

was the same in the two cases. If you put an obstacle in the path of each body so as to stop both in the same *time*, they would each deal the same blow. Suppose, for instance, that the cannon-ball and the truck were to meet each other end-on, and the ball were to remain imbedded in the material of the truck, both would be stopped dead by the impact.

39. Before passing on to the action of force on matter, it will be well to explain that now we have come to deal with the motion of actual pieces of matter, we shall, if we wish to consider a piece so small that its parts may be neglected, use the term *particle* instead of 'point;' meaning by *particle* a point possessing inertia, or a *material point*. A 'particle' may have any finite *mass*; its *size*, indeed, is to be small (or at any rate negligible), but its *density* may be anything—infinite if we like. A body whose parts are taken into account may be called an 'extended body,' but if stress is wished to be laid on the fact that these parts are immovable relatively to each other, it will be called a *rigid body*. An extended body whose parts are capable of relative motion is either an elastic or else a plastic body. (Chapter X.)

Also it will be well to point out that the parallelogram and polygon laws apply to the composition of momenta just as they do to the composition of velocities (sect. 27). For the momentum of a given mass is simply proportional to its velocity, and the resultant velocity of a particle when multiplied by its mass must be its resultant momentum.

(II.) SPINNING MOTION OF AN EXTENDED BODY, OR ROTATION.

(*Moment of Inertia and Moment of Momentum.*)

40. We have already partly seen (sect. 17) that when we come to consider the motion of a rotating body, the distance of each particle from the axis of rotation is always coming in as a factor, multiplying the term which previously had

been sufficiently expressive. As this product so often occurs, it is convenient to have a name for it, and the name employed is *moment*. The moment of any physical quantity is the numerical measure of its importance. [This must not be confounded with *momentum*, with which it has nothing to do.]

So the actual velocity of a particle of a rotating body might be called the *moment* of the angular velocity, for it equals ωr, r being measured from the particle to the axis of rotation. The actual *acceleration*, again, is the *moment* of the angular acceleration—that is, it equals αr (see sect. 18).

It often happens that the distance from the axis comes in as a factor *twice*, so that we have a moment of a moment, which is called a second moment.

Thus for some purposes it is convenient to speak of the moment of the velocity of a particle of a rotating body—that is, vr; and this is the *second* moment of its *angular* velocity, being equal to ωr^2. The moment of *momentum* of such a particle is, of course, mvr, or, as it may also be written, $mr^2\omega$.

41. These terms being understood, we will proceed to consider how we must define the quantity of motion of a rotating body, or a system of circularly moving particles. Simple momentum, or product of velocity and quantity of matter, will not do, for the effect produced by a given shock depends not only upon this, but also upon how far distant from the axis the bulk of that matter is. For consider a flywheel; which you know is a large heavy wheel fixed to the crank-shaft of stationary engines and driven at a high speed, not for the purpose of communicating its motion to a lathe-band or anything, but simply for the purpose of storing up a certain quantity of motion sufficient to carry the engine over its 'dead points,' and also over any accidental shocks or sudden impediments which the machinery may experience: it is made massive so as to have

great inertia, it is also made to go fast so that its parts may possess great momentum; but besides this it is made large, and nearly all the mass is placed in its rim, which not only increases the momentum but causes that momentum to have a great *leverage;* so that altogether the motion stored up in the wheel has a great moment of momentum.

For just as the power or moment of a force depends not only on its magnitude but also on the place at which it is applied—not only on its strength but on its leverage—being equal to the product of the force into its distance from the fulcrum (for example, the longer a crowbar is, the more power it gives you; the more unequal the length of the arms of a steelyard, the bigger the weight which can be balanced by a little one; and so on, see sect. 148); so with the flywheel, the effect or power of its stored-up motion depends not only on the actual quantity of motion or momentum of the rim, but also on the distance this rim is from the axle—that is, on the radius of the wheel. It depends, in fact, on the *moment* of its momentum, Mr.

42. Now if the wheel were a simple infinitely thin rim, the meaning of this would be simple enough; r would stand for the radius of the rim, and M for the product of its mass and velocity, mv (sect. 37); but any actual wheel must have a rim of some thickness, as well as some spokes and a nave, so the meaning of neither M nor r is quite clear without further definition.

The moment of momentum of a rotating body is the sum of the moments of momenta of its several particles.

Let a wheel turn with the uniform angular velocity ω. A particle of mass m_1, at a distance r_1 from the axis, and moving with velocity $r_1\omega$ or v_1, has a momentum m_1v_1, and therefore a *moment* of momentum $m_1v_1r_1$, or what is the same thing, $m_1r_1^2\omega$. Similarly with a particle of mass m_2 at a distance r_2; and with one of mass m_3 at distance r_3, and so on; hence the *moment of momentum* of the whole

wheel is the sum of these terms for all the particles in the body,

$$m_1v_1r_1 + m_2v_2r_2 + m_3v_3r_3 + \dots\dots\dots$$

or, as it is often written, $\Sigma\,(mvr)$.

Since $v = r\omega$, and since ω is the same for every particle as for the whole body, we may write the above expression for the moment of momentum in this equivalent form,

$$\omega(m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \dots\dots\dots) = \omega\Sigma(mr^2)\;;$$

or in words, the moment of momentum of the wheel is the angular velocity multiplied by the sum of the second moments of inertia of every particle in the wheel.

In this last form, $mr^2\omega$, the moment of momentum is often called the *angular* momentum; because, instead of being simply the product of inertia and velocity (as momentum is), it is the product of a *moment* of inertia and *angular* velocity.

43. In the last paragraph we have the occurrence of the *second* moment of mass or inertia, mr^2, and indeed this occurs in Dynamics so much more frequently than the first moment (mr), that it is usually called *the* moment of inertia.

The moment of inertia of any rotating body about its axis of rotation is the sum of the second moments of the masses of all the particles in it about that axis; and we will denote it by I, so that $\mathrm{I} = \Sigma(mr^2)$.

The angular momentum, or moment of momentum, of the above flywheel is thus simply $\mathrm{I}\omega$.

The value of the moments of inertia of bodies of regular shape is obtained by actual calculation of the above sum, $m_1r_1^2 + m_2r_2^2 + \dots\dots$, for any required axis. The process is easy to those who have learned how to integrate: to others the following list of results may be useful:

LIST OF MOMENTS OF INERTIA.

A. *About an Axis of Symmetry.*

(1) For a thin ring or hollow drum (of mass m and radius r)..mr^2.

(2) For a thick ring or drum of internal radius r_1 and external r_2 .... $\frac{1}{2}m(r_1^2 + r_2^2)$.
(3) For a solid disc or cylinder of any length .... $\frac{1}{2}mr^2$.
(4) For a solid sphere .... $\frac{2}{5}mr^2$.
(5) For a thin hollow sphere .... $\frac{2}{3}mr^2$.
(6) For a rectangular plate or bar of length l, breadth b, and any thickness in direction of axis .... $\frac{1}{12}m(l^2 + b^2)$.
(7) For a thin rod about middle .... $\frac{1}{12}ml^2$.

B. *About other Axes.*

(8) About any axis parallel to an axis through the centre of gravity, but at a distance a from it, the moment of inertia = the central value + ma^2. For instance (see No. 7):
(9) For a thin rod about one end .... $\frac{1}{12}ml^2 + m(\frac{1}{2}l)^2 = \frac{1}{3}ml^2$.
(10) The moment of inertia of an isosceles triangular area about its median line, (base = b) .... $\frac{1}{24}mb^2$.
(11) The moment of inertia of a triangular area of height h, about its base as axis .... $\frac{1}{6}mh^2$.
(12) The moment of inertia of any plane lamina or plate about two axes through its centre of gravity, at right angles to each other and to the axis of symmetry, add up to that about the symmetrical axis. Thus: for a circular plate spinning about a diameter, the moment of inertia is (see No. 3) .... $\frac{1}{4}mr^2$.
(13) The moment of inertia of a rhombus about a normal axis through its middle (its longest diagonal being l and its shortest b) .... $\frac{1}{24}m(l^2 + b^2)$.

EXAMPLES—VII.

(1) Find, in foot and pound units, the moment of inertia of a rod of iron 5 feet long and 1 square inch in cross-section, about an end, given that the mass of a cubic foot of iron is 480 lb.
(2) Find the moment of inertia of the rim of an iron flywheel of 11 feet mean diameter, the cross-section of the rim being 12 square inches.
(3) Find the moment of momentum of the above rim when the wheel is making 200 revolutions a minute.

(4) Find the moment of inertia of a thin rectangular plate measuring 4 feet by 3 feet and weighing 5 lb.:
   (*a*) About an axis through its centre normal to its plane.
   (*b*) About a parallel axis through one corner.
   (*c*) About a long edge.
   (*d*) About a short edge.
   (*e*) About a median line drawn lengthways.
   (*f*) About a median line drawn breadthways.

(5) Find the moment of inertia of a hollow sphere 50 centimetres in diameter, weighing half a kilogramme, and spinning like a teetotum.

(6) Compare the moments of inertia of a hollow sphere and a hollow cylinder of the same diameter and weight, also of a solid cylinder and a solid sphere of the same diameter and weight.

(7) Find the moment of inertia, about its point of suspension, of a solid sphere 1 foot in diameter and weighing 50 lb., swinging at the end of a string 6 inches long.

(8) Find the moment of inertia of a triangular iron plate, $\frac{1}{4}$-inch thick, about its base; the base being 5 inches long, and the height 6 inches.

(9) Find the moment of inertia of the same plate about an axis passing through its centre of gravity and parallel to the base.

In the above list of Moments of Inertia,

(10) Show that No. 13 can be deduced from Nos. 11 and 12.

(11) Deduce No. 10 from No. 11.

(12) Deduce (2) from (3), and (1) from (2).

(13) Deduce (5) from (4); the volume of a sphere being $\frac{4}{3}\pi r^3$, and its surface being $4\pi r^2$.

CHAPTER IV.

ON FORCE AND MOTION (*Dynamics*).

It was stated in the Introduction that force produced two kinds of effects on matter—'acceleration' and 'strain.' In the present chapter we will consider only the first or *motive effects of force*—that is, the effects of force on rigid *bodies or particles (see sect. 39); and first on particles* moving in the direction of the force—

(I.) ON THE SPEED OF MOTION AS AFFECTED BY FORCE; OR, FORCE AND RECTILINEAR MOTION.

(*Dynamics of a Particle.*)

44. When a single force F is applied to a certain quantity of matter or mass, m, for a given time, a certain quantity of motion or momentum is generated in the mass. If the same force (for example, a piece of elastic stretched to the same extent as before) is applied to a greater quantity of matter for the same time, it will move with less velocity, but the *product* of the quantity of matter and the velocity—that is, the *quantity of motion* or the *momentum*—will be found to be the same; so the force may be measured by the momentum generated by it per second, since this is constant, and depends on nothing but the force. If the same force be applied for a greater time, a proportionally greater quantity of momentum will be generated; hence the measure of the *force* is the momentum generated per second, and is obtained by dividing the whole momentum generated by the time taken to do it; or in symbols,

$$F = \frac{mv}{t};$$

and the *unit* force will be that which can generate unit momentum in unit time.

45. Force, then, by this definition comes to be *rate of change of momentum*, just as acceleration was defined to be rate of change of velocity:

$$a = \frac{v}{t} \text{ (sect. 10).}$$

Hence force bears the same relation to acceleration as momentum does to velocity: each, in fact, equals the other multiplied by m, or

$$F = ma.$$

This last is a very convenient form of the definition, and may be expressed thus:

A force is numerically equal to the acceleration it can produce in unit mass; and in general a resultant or unbalanced force is measured by the product of the mass moved into the acceleration produced, being proportional to the two conjointly; or concisely,

FORCE = MASS ACCELERATION.

This is, indeed, the fundamental relation of Dynamics, for it makes all that we have learned about motion in the abstract (Kinematics) available for dynamical problems—that is, for all problems involving force. It is called Newton's SECOND LAW OF MOTION.

46. The unit of force may be expressed in these three different but equivalent ways:

The unit of force is that which causes unit acceleration in unit mass (one pound or one gramme of matter);

Also, unit force is that which generates unit momentum in unit time, as said above;

Also, it is that which, acting on unit mass for unit time, causes it to move with unit velocity. So, if the British unit of force act on a pound for a second, the pound at the end of that second will be moving at the rate of one *foot*

per second. If the C.G.S. unit of force acts on a gramme, the gramme will be moving at the rate of n centimetres per second after the lapse of n seconds.

It is often convenient to have a name for the unit of force as defined in any of these equivalent ways. The name *poundal* has been suggested for the British unit of force in order to indicate a connection between it and the British unit of mass (not by any means to signify that the force unit equals the *weight* of a pound: it is nearer the weight of half an ounce). A poundal is also called the *British* absolute unit of force, to distinguish it from the unit founded on the metric system, which involves *grammes* and *centimetres* instead of pounds and feet. This C.G.S. unit of force is now very frequently called a *dyne*. It is, of course, that force which, acting on a gramme for a second, generates in it the velocity of one centimetre per second. It is a very small force indeed, only about the thousandth part of the weight of a gramme, which is itself only about 15 grains. One poundal equals 13825·38 dynes. A poundal equals a pound foot per second per second, or briefly $\frac{\text{lb. foot}}{\text{sec.}^2}$. A pound weight is about 32 poundals (see next chapter); a gramme weight is about 981 dynes. These standard weights are frequently used as practical units of force, and in statical problems are very convenient. The load of a ton, a hundredweight, or a kilogramme, is an easily imagined quantity, and forces so expressed are said to be stated in gravitational measure, since weight depends on the earth's gravitative attraction. There is no difficulty in translating these gravitational units into absolute* units whenever the problem ceases to be statical and we need to enter on dynamical considerations.

* The word 'absolute' is not very appropriate in this connection, but it is constantly so employed. The meaning to be expressed is that the unit is a *completely specified* one, not depending on the properties of any concrete piece of matter. For instance, the pull of a spring, stretched by a certain amount, might

The dyne being a very small unit, a megadyne (or million dynes) is often employed. A kilogramme weight is nearly equal to it, being equal to 981,000 dynes; or 2 per cent. short of the full megadyne. A pound weight is 11 per cent. less than half a megadyne (being about 445,000 dynes).

The Gravitation Constant.

But though the force of gravitation thus appears to be so great, this is only because the earth is so massive. Newton found that the same force exists between all material bodies, that it is proportional to the masses, and that, if the masses are spherical, it is inversely proportional to the square of the distance between their centres. These facts may be expressed by the equation $F=\gamma\frac{mm'}{d^2}$, where γ is a constant, called the gravitation constant, whose value must be determined by direct observation or experiment. The result of such experiments is that, very approximately,

$$\gamma=\frac{666}{10^{10}}\frac{(\text{cm.})^3}{\text{gm.}(\text{sec.})^2}=\frac{1}{10^9}\cdot\frac{(\text{ft.})^3}{\text{lb.}\ (\text{sec.})^2},$$

which may serve as a memorandum till it is intelligible. The gravitation force between ordinary pieces of matter is so small that if a couple of lead globes, each weighing a pound, were placed with their centres one foot apart, their mutual attraction would be only a thousand-millionth part of a poundal; so that, if they were perfectly free to move in an undisturbed way, they would each have moved three-quarters of an inch in 3 hours—that is, they would be an inch and a half nearer to each other, after the lapse of 3 hours, under the influence of their mutual attraction.

be used as a unit or standard of force, but it would be by no means an *absolute* one. So also, the weight of a pound is not an absolute unit, for it depends on the neighbourhood of the earth.

47. The fundamental connection between force and acceleration, $F = ma$ (sect. 45), may be written, of course, in two other forms; and this one,

$$a = \frac{F}{m},$$

is an abbreviated statement of the fact that when a force F acts on a mass m, the acceleration produced in it is the ratio of the force to the mass.

Let us take an example to illustrate the application of this. Find the distance travelled in 8 seconds by a mass of 2 lb. which starts from rest, and has a force of 6 poundals acting on it all the time.

The acceleration, or velocity acquired per second, is

$$a = \frac{F}{m} = \frac{6 \text{ poundals}}{2 \text{ pounds}} = 3 \text{ F.P.S. units.}$$

The whole velocity acquired in the 8 seconds is therefore 24, and hence the average velocity is 12 feet per second. The distance travelled is the average velocity multiplied by the time, that is, 96 feet; which is the answer.

Or we might, without troubling about the velocity, have applied the formula $s = \frac{1}{2}at^2$ as soon as we knew the value of the acceleration, $a = 3$, and of course we should have arrived at the same result. But all this latter part is simple Kinematics: the only dynamical part was the finding of the acceleration from the given force and mass—namely, $a = F \div m$.

Whether the body is in motion or not when the force begins to act, matters nothing—the acceleration produced is precisely the same. Of course the distance travelled in a given time will be different, because of the initial velocity, (v_0t will have to be added to the $\frac{1}{2}at^2$); but all that was considered in Kinematics, Chapter II.

48. The following is Newton's statement of the above connection between force and motion:

'*Change of motion is proportional to impressed force, and takes place in the direction in which the force acts;*'

Or as it has been restated by Professor Clerk Maxwell, in equivalent modern language:

'The change of momentum of a body is equal to the impulse which produces it, and is in the same direction.'

By *impulse* is meant the *product* of the *force* acting, and the *time* it lasts; for it is on both these that the motive power of a force depends. Thus the blow of a hammer is a very great force while it lasts; but as it is only momentary, its *impulse* (or motive effect) may not be so great as a much smaller force applied continuously for some time * (cf. sect. 38). The motive effect or impulse is proportional both to the strength of the force, F, and to its duration, t; and hence it is defined as the product Ft. So the first portion of the above statement is, in symbols,

$$mv = \mathrm{F}t;$$

where v represents the velocity gained by the mass m owing to the action of the force F for a time t; it is, in fact, simply the fundamental relation of sect. 44 in another form.

It is convenient at this stage to refer to the numerical examples given in sect. 38, and to realise that to start the cricket-ball with momentum 14 F.P.S. units would require a force of 14 poundals lasting for a second, as when thrown; or a force of 14,000 poundals (say 2 tons) lasting the thousandth part of a second, as when struck with a bat. Also, that to start the cannon-ball in the thousandth part of a second would require a force of 80 million poundals, which is about equal to the weight of a thousand tons; a force which, applied to the sectional area of such a cannon-ball (say 33 sq. inches), would need an average pressure in the gun of 30 tons to the square inch. All this on the assumption that the thousandth part of a second represents fairly the time between the ignition of the powder and the ejection of the ball.

The railway truck if pushed by three men each exerting a force of one hundredweight, would get up its given speed in

* This is best observed by first striking sharply, and then pushing steadily, a thing on wheels where the friction is small. The advantage of a blow is felt, not when you want to move a massive body, but when you have a great force of friction to overcome, as in hammering a nail.

$80640 \div (336 \times 32)$—that is, in about $7\frac{1}{2}$ seconds; supposing there was no friction and that the line was perfectly level.

As for the jet of water, it represents a momentum of ten million C.G.S. units per second, and hence corresponds to a steady force of ten million dynes. This force must be operating, to propel the jet, over the sectional area of the nozzle; and if this area is a circle half a centimetre in diameter, the pressure to which the water is pumped must be about 50 megadynes per square centimetre—that is, about 50 atmospheres, or 700 lb. to the square inch.

49. The above statement, in italic or black type (sect. 48 and sect. 45) is often called, as has been already stated, the second law of motion: it might with propriety be called *the* law of motion, or the law of force and motion. It is very general, and involves a great deal. *It is the fundamental law of mechanics.*

First, it shows that where there is no force there is no change of momentum—that is, that a body not acted upon by any external force, if in motion, will continue with that motion unaltered, and, if at rest, will remain at rest; a fact often stated separately as the law of inertia, or the first law of motion (sect. 33).

It further declares implicitly that if a force act on a body in motion, it produces just the same effect as if it had acted on the same body at rest—that is to say, the *state* of the body on which the force acts is immaterial, as nothing is said about it in the statement.

Moreover, the second law implies that if two or more separate and independent forces act on a body, each produces its own change of motion in its own direction without regard to the others.

50. This last is an important aspect of the law, and tells us that the operation of compounding together a lot of independent forces is just the same as that of compounding together the motions which each force separately tends to produce in the same time.

Thus if AB represents the quantity of motion (that is, the momentum) which would be produced by one force by itself in a second, and BC the motion which would be produced by another force by itself; then AB and BC may also be taken to represent the two forces themselves. But we learn from Chapter II., and from sect. 39, that the resultant of the two motions AB and BC is the single motion AC, hence AC may be taken as representing the *resultant* force—that is, a force which, if acting by itself, would produce precisely the same effect as the other two forces acting together; provided they are independent of each other.

Hence all that we have said about the composition of motions applies equally well to the composition of independent forces. In other words, forces are compounded by the parallelogram and polygon laws just as motions are compounded (see sect. 24 and Chapter VII.).

51. Moreover, we learn that in order to specify the translating power of a force, it is only necessary to specify the velocity it is able to produce in unit mass in a second; which is readily done by drawing a straight line anywhere of definite length in a definite direction. But we shall soon learn (sect. 55) that, as force has *rotating* as well as *translating* power, it is necessary, for the complete specification of a force, to assign also its position or line of action; it is not necessary to assign it any definite place in that line.

Hence three things determine a force—Direction (with sign), Position, and Magnitude. As these things are possessed by an arrow-headed line of given length, such a line is often used to symbolise a force. This ↓, for instance, would be one force, and this → a force of the same magnitude as the first, but in a different direction; while this other one, equal and parallel to the first, ↓ would be equivalent to the first in translating power, for it has the same magnitude and direction, but different in rotating power, having a different position, that is, line of action. The only

defect of this mode of representation is that it is a little too expressive—that is, it expresses a little more than is wanted. For → and →, though two distinct *lines*, represent the same *force* in every respect, having the same direction, magnitude, and line of action—the rotating and translating powers are the same (see end of sect. 57). For further development of this, see Chapter VIII.

52. There is one more thing about force which is very important, but in the present stage its full meaning can scarcely be appreciated, and that is the fact, mentioned in the Introduction, that force is always due to the *mutual* action of two bodies or systems of bodies; that *every* force, in fact, is one of a pair of equal opposite ones—one component, that is, of a *stress*—either like the stress exerted by a piece of stretched elastic, which *pulls* the two things to which it is attached with equal force in opposite directions, and which is called a *tension*; or like the stress of a pair of compressed railway buffers, or of a piece of squeezed india-rubber, which exerts an equal *push* each way, and is called a *pressure* (see sect. 3). Newton's law concerning this is what is called his THIRD LAW OF MOTION:

'*Reaction is always equal and opposite to action—that is to say, the actions of two bodies upon each other are always equal and in opposite directions.*'

This may be called the law of stress, and it has been shown by Professor Tait to be susceptible of considerable development (see Thomson and Tait's *Natural Philosophy*, art. 269, and see also Chapter VI. of the present text-book). It is deducible from the first law of motion (see Maxwell, *Matter and Motion*, art. lviii.), for if the forces exerted by two parts of the same body on each other were not equal and opposite, they would not be in equilibrium; and consequently two parts of the same body might, by their mutual action, cause it to move with increasing velocity for ever, the possibility of which the first law denies. The

same proof holds without modification for the mutual forces between any two or more bodies; for those bodies may be regarded as a single system or complex body, within which all internal forces must balance, else would there be the impossible result of an internal force capable of accelerating the system.

We have already shown (sect. 49) that the first law is a special case of the second, and now we have deduced the third from the first; hence all are really included in the second, which is therefore excessively important.

EXAMPLES—VIII.

(1) What is the acceleration when a force of 36 units acts on a mass 4; and how far will the mass move in 10 seconds?

(2) What is the least force necessary to cause 15 lb. to move 30 feet from rest in 5 seconds?

(3) If a mass of 7 lb. is acted on by two opposite forces of magnitudes 56 and 42 respectively, what is the acceleration; and what will be the momentum generated in 5 seconds?

(4) How long must a force of 8 poundals act on a 20 lb. mass to change its velocity from 2 to 26 feet per second?

(5) In what distance will a force of 2 poundals be able to stop a mass of 30 lb., which at the time the force begins to act is moving 50 feet every second?

(6) A mass of 20 lb., which has been going 40 feet per second, is now retarded by a constant force equal to the weight of 4 pounds. How soon will its velocity be 24 feet per second in the reverse direction?

(7) Find the force which, acting on a mass of 2 kilogrammes, gives it a velocity of 98·1 cm. per second in 5 seconds. Compare this force with the weight of the body.

(8) If a mass of 6 lb. is propelled so as to gain a velocity of 10 feet a second every second, what is the magnitude of the force urging it?

(9) If a mass of 6 lb. be pushed by a force of 2 poundals without friction for 5 minutes, how much will the momentum of the mass be altered?

(10) If a certain force acting on a mass of 6 lb. for 4 seconds

gives it a velocity of 40 feet per second, through what distance would the same force move a mass of 10 lb. in 5 seconds?

(11) What weight is equal to a force of 1200 absolute foot-pound-second units?

(12) What weight is equal to a force of a million dynes? How many dynes will support an ounce against gravity? How many dynes can support a gramme? How many correspond to the weight of a ton?

(13) A force equal to the weight of a cwt. acts on a ton for 2 minutes. What velocity will it produce? If a force equivalent to the weight of a ton operated on a quiescent hundredweight for 2 minutes, how far would it push it?

(14) If a force equal to the weight of a gramme pull a mass of 1 kilogramme along a smooth level surface, find the velocity when the mass has moved 1 metre.

(15) A mass of 100 grammes acquires a velocity of 30 cm. per second in 10 seconds. Find the force acting on it.

(16) A load of 50 lb. is being lowered by a cord from a height. Find the tension in the cord:

(*a*) When the speed is increasing at the rate of 8 feet-per-second per second;

(*b*) When it is decreasing at the same rate;

(*c*) When the speed is uniform.

(17) What steady force must act on a mass of 10 lb. initially at rest in order to move it 144 feet in 3 seconds? How does this force compare with that of the earth's attraction for the mass?

Impact.

53. When two or more bodies in free motion impinge on one another, their action and reaction are equal and opposite, or, in other words, there is no outstanding or resultant force tending to move the system of bodies as a whole in any direction. Whatever was their average motion before impact, that same will continue to be their average motion after impact. The momentum of the whole can only be changed by a force exerted by something outside the system; internal forces can rearrange the distribution of momenta,

but are incompetent to affect total momentum. Hence, m_1 m_2 being the masses, if u_1, u_2 are their initial velocities, and v_1, v_2 their final velocities, the above deduction from Newton's third law may be written :

The resultant of m_1u_1 and m_2u_2 = the resultant of m_1v_1 and m_2v_2; that is, the initial momentum and the final momentum are equal in both magnitude and direction.

The impact of particles or homogeneous spheres may be either direct or oblique. The impact is direct when they approach each other along the same line and when their surfaces at the point of contact are perpendicular to this line, so that the bodies also recede along the same line after impact. In this case the equation of momenta is

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2.$$

The line through the point of contact perpendicular to their surfaces at that point may be called the line of the blow, or the line of impact. If the velocities of the approaching bodies are inclined to this line, they may be usefully resolved into components—one along this line, and the other at right angles to it—the first being the component in the direction of impact, and the other the transverse component of the velocity of each body. The direct components of momentum obey the above law. The transverse momenta not only obey the same equation of total equality, but are individually absolutely unchanged by the impact.

Another mode of stating the law of constancy of total momentum is to say that the motion of the centre of gravity of the two bodies continues unchanged by the impacts, or by any other exertion of internal forces. For, as will be shown in sect. 131, the centre of gravity of the two bodies is a point between them such that its distance, x, from any line in the same plane with them is given by

$$(m_1 + m_2)x = m_1x_1 + m_2x_2,$$

whence it follows that its velocity, u, perpendicular to that same line, is given by $(m_1 + m_2)u = m_1u_1 + m_2u_2$ before im-

pact, and that its velocity, v, after impact, is given by $(m_1+m_2)v=m_1v_1+m_2v_2$. Hence, taking the line of reference perpendicular to the line of impact, that is, attending to the component velocities in this line, and remembering that $m_1u_1+m_2u_2=m_1v_1+m_2v_2$, we see that $v=u$, so that this component of the velocity of the centre of gravity is unchanged by the impact; and, *à fortiori*, the transverse component is unchanged, therefore the motion of the centre of gravity is entirely unaffected by the blow.

And this is general: for instance, when a shell explodes in mid-air, the centre of gravity of the whole of its materials continues its parabolic orbit unaltered until some of the pieces strike external objects. Or, again, when a shot is fired from a gun, the total momentum after explosion is the same as it was before—namely, 0: that is, the gun recoils with a momentum which is equal and opposite to that of the shot and the powder gases forward. This fact is utilised in a rocket in an emphatic manner.

All this represents the first fundamental law of impact or of internal forces in general, and it is called the law of 'conservation of momentum.' It is to be entirely distinguished from the conservation of energy; indeed, energy is by no means obviously conserved in cases of impact. Heat and sound, as well as rotation, have to be taken into consideration before the conservation of *energy* can be asserted (see Chapter VI.), but the conservation of momentum is a simple variant or extension of Newton's first law of motion.

The next fact that has to be stated is of a more empirical character—namely, that, in the case of direct impact, the relative velocity of recoil always bears a fixed ratio to the relative velocity of approach for a given pair of bodies, provided that, in general, the shock is not so great as to permanently deform or break them.

Consider two spheres moving in the same line and one

overtaking the other. Before impact their relative velocity is $u_1 - u_2$; after impact their relative velocity is $v_2 - v_1$; and these two velocities are proportional, so that the ratio $\frac{v_2 - v_1}{u_1 - u_2}$ = a constant. This constant, which is usually denoted by e, is called the 'coefficient of restitution,' which we may shorten into *the recoil-ratio* of the two bodies.

In a few simple cases, such as perfectly elastic spheres or equal rods impinging 'end-on,' this coefficient is practically unity; but in general it is less than 1 by reason of the setting up of rotations, and also by reason of some energy taking the form of vibrations, whether of sound or heat; and in the extreme case of perfectly inelastic bodies, like putty, or dough, or wet clay, the value of the coefficient is zero. In these latter cases the bodies move on together after impact without any recoil. The following four statements are now easy to verify for cases of direct impact; though, like the rest of this section, they will not be found easy till more has been read:

(1) Two equal bodies with $e = 1$ interchange their original velocities after impact. (For example, if one had been stationary at first, the other will be left stationary at last.)

(2) When $e = 1$, the initial and final *energies* are equal.

(3) When $e = 0$, the loss of visible energy at impact is

$$\tfrac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right)(u_1 - u_2)^2 = \tfrac{1}{2}M(u_1 - u_2)^2,$$

$$\text{where } \frac{1}{M} = \frac{1}{m_1} + \frac{1}{m_2};$$

the loss being equal to the energy of an imaginary particle moving with the relative velocity of approach of the two impinging bodies, and of mass equal to half their harmonic mean.

(4) In all cases of direct impact the loss of energy is $1 - e^2$ times the above amount.

Oblique Impact.—If the impact is oblique, the component velocities in the direction of impact are changed in accordance with the above equations—namely:

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \quad (1)$$
$$\text{and } v_2 - v_1 = e(u_1 - u_2); \quad (2)$$

while their components perpendicular to this direction are, as previously stated, wholly unaffected.

If one of the masses (m_2) is infinite, equation (1), though still true, is useless, but equation (2) is then sufficient. Moreover, since the velocity u_2 of the infinite mass is practically unaltered by the collision, and will most naturally be taken as zero, the equation reduces to

$$v_1 = -eu_1,$$

that is, the component velocity of m_1 in the direction of impact is reversed and diminished in the ratio of $e:1$. The transverse component is, of course, unaltered by the collision.

It is interesting to note that, since the path of the centre of gravity of two colliding masses is absolutely unaltered by the collision, the circumstances of impact are the same as if the two masses each collided with an infinite mass moving with the velocity of their centre of gravity. If we consider their motion relative to their centre of gravity (obtained by compounding with the actual velocity of each mass a velocity equal and opposite to the velocity of their centre of gravity), the component velocity of each in the direction of impact is reversed and diminished in the ratio of $e:1$, in accordance with what always happens when a finite mass collides with an infinite mass; and their transverse relative velocities continue unchanged. These ideas will assist us to represent in a diagram the change of velocities caused by a collision under given conditions.

The following general construction can be given, and its proof left as an exercise. If AO represents in magnitude and direction

the velocity of some particle A, and BO the velocity of any other particle B, and if G is the centre of gravity of the two particles A and B, then GO is the velocity of their centre of gravity; and AG, BG are the velocities of the particles relative to their centre of gravity. After impact has taken place, in some line to which MN is drawn normal, the velocity of the centre of gravity continues unaltered, but the velocities of the particles become A′O and B′O respectively, such that, if $e=1$, the line MN is equally inclined to AB and to A′B′, while A′G = AG and B′G = BG.

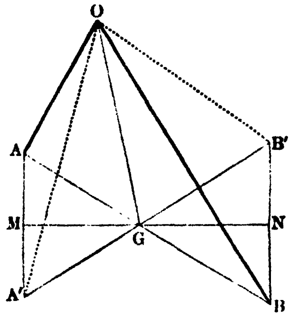


Fig. 12.

(If e is not unity, then the tangent of the angle between A′B′ and MN is e times the tangent of the angle between AB and MN.)

It should be noticed that AM, BN are the *direct* components, and MG, NG the transverse components, of the velocities AG, BG; consequently in all cases MG, NG remain unchanged, while the direct components are reversed and diminished in the ratio $e:1$, so that $A'M=-e\cdot AM$, and $B'N=-e\cdot BN$. (In the figure, $e=1$.)

EXAMPLES—IX.

(1) Two balls of masses 4 lb. and 8 lb. are moving towards each other with velocities 25 and 3 feet per second respectively. After impact they move on together. Find the common velocity.

(2) Two perfectly inelastic bodies of masses 120 and 150 lb. respectively, moving with equal opposite velocities of 18 feet a second, impinge directly on each other. Find the subsequent velocity.

(3) A lump of clay when thrown horizontally against a mass 20 times as great, resting on perfectly smooth ice, makes it move 3 feet in 2 seconds. What speed had the clay at the moment of impact?

(4) Two masses of 6 and 10 lb., moving in opposite directions with velocities of 10 and 8 feet per second respectively, collide. If the velocity of the smaller mass be exactly

reversed by the impact, find the coefficient of restitution for the two bodies.

(5) Find the velocities of two bodies after collision, their masses being 18 lb. and 32 lb., their velocities (in the same straight line and in the same direction) 12 and 7 feet per second, and their coefficient of restitution $\frac{1}{4}$.

(6) A 1-oz. bullet fired from a 20-lb. rifle pressed against a mass of 180 lb., kicks the latter back with an initial velocity of 6 inches per second. Find the bullet's initial velocity.

(7) A bullet of mass $\frac{1}{10}$ lb. is fired with velocity 300 feet per second into the middle of a wooden block weighing 30 lb. hung by a very long cord. What is the common velocity of pendulum and bullet just after the collision?

(8) Two masses of 10 and 20 lb., moving in the same direction with velocities of 25 and 15 feet per second respectively, collide. Find their velocities after impact, assuming them to be perfectly elastic.

(9) Masses of 4 and 6 lb. collide when moving in opposite directions with velocities of 8 and 12 feet per second respectively. If their coefficient of restitution be $\frac{1}{4}$, find their velocities after impact.

(10) Two bodies, for which the coefficient of restitution is $\frac{1}{4}$, approach each other with equal velocities of 20 feet per second. After collision one body, the mass of which is 5 lb., returns with velocity 15 feet per second. Find the mass of the other body, and its velocity after collision.

(11) Two masses of 5 and 3 lb. collide when moving in opposite directions with velocities of 4 and 10 feet per second respectively. If the smaller is just brought to rest by the impact, find the coefficient of restitution for each body.

(12) A bullet, weighing 50 grammes, is fired into a target with a velocity of 500 metres a second. The target is supposed to weigh a kilogramme, and to be free to move. Find, in kilogrammetres, the loss of energy in the impact.

N.B.—Remember that $\frac{1}{2}mv^2$ gives a result in *absolute*, not in *gravitational*, units.

(13) A ball let fall on a stone slab from a height of 16 feet bounces the first time to a height of 9 feet. What is the coefficient of restitution, and how high will the ball bounce next time? Find also the total distance it will travel before coming to rest.

(14) Two smooth spheres whose masses are proportional to 10

and 8, moving in directions perpendicular to each other, with velocities of 20 and 30 feet per second respectively, collide so that the line of centres makes angles of 30° and 60° respectively with the directions of the motions. Calculate their velocities after impact; taking the recoil ratio as unity. Also find them by means of a diagram.

(15) A cannon on an armoured train fires a 10-lb. projectile rearwards with a velocity of 2000 feet a second. If the truck with the cannon weighs 2 tons, what initial velocity is imparted to it?

(16) A Maxim gun delivers 300 one-ounce bullets per minute with a speed of 1600 feet a second. What force is necessary to hold the gun still?

(17) A jet of water from a fire-engine delivering 30 gallons of water per minute through a quarter square inch opening impinges directly on a brick wall. What force does it exert?

(18) Why does a foul gun kick worse than a clean one, notwithstanding that it ejects the shot with less velocity?

(II.) ON ANGULAR VELOCITY AS AFFECTED BY FORCE; OR, FORCE AND ROTATION.

(Dynamics of a Rigid Body.)

54. When force acts on an extended piece of matter, it produces in general both motion and strain (sect. 5). The latter we do not want to consider at present; so to exclude it, we suppose the body to be *rigid*—all its parts rigidly bound together and incapable of distortion or relative displacement. The effects of force on such a body are translation and rotation. If the effect is translation only, the body acts like a particle (sect. 39), as if all its mass were concentrated at a point (called its **centre of inertia**, or sometimes its centre of gravity), and the second law of motion as stated for particles applies to the rigid body; so that if R is the resultant of all the external forces acting on the body, and if m is its mass, the acceleration of its centre of inertia is $\frac{R}{m}$. This is true anyhow, whether there is rotation or not;

but when rotation is allowed the subject becomes much more complicated, especially if translation is possible as well. We can, however, consider rotation by itself, by supposing one line or point in the body to be fixed in position, so as to constitute an axis or centre of rotation.

55. All we can say about the subject here is, that in estimating the rotating effect of a force, one must not only consider its impulse—that is, its magnitude multiplied by its duration (sect. 48)—but we must also consider its position; how far its line of action is from the fixed line or axis of rotation: the farther it is off, the more effect it has; its *moment* (sect. 40) is greater.

Suppose a force acts on a body only capable of rotation, at a distance R from its fixed axis: the *moment of momentum*, or *angular momentum*, generated [$\Sigma(mvr)$, or $\Sigma(mr^2\omega)$, see sect. 42], equals the product of the moment of the force, FR, into the duration, t; in other words, it equals the *moment of the impulse* $Ft\,.\,R$.

This is expressed by the following equation, where the *moment of inertia* of the body $\Sigma(mr^2)$ is denoted by I (see sect. 43):

$$I\omega = FRt,$$

or, *moment of momentum* = *moment of impulse*,

which is an extension of the simpler particle equation, (sect. 48), *momentum* equals *impulse*, $mv = Ft$.

56. This equation may also be written (since $\omega = at$),

$$\text{angular acceleration} = a = \frac{FR}{I} = \frac{\text{moment of force}}{\text{moment of inertia}},$$

which is evidently analogous to the simple, and, for particles, fundamental equation (sect. 47),

$$\text{acceleration} = a = \frac{F}{m} = \frac{\text{force}}{\text{inertia}},$$

and includes it as a special case.

For an application of this equation, see sect. 142. Read again, carefully, sections 40–43.

Moment of a Force.

57. The idea of the moment of a force is a very important one, and will occur again and again in Statics (Chap. VIII.). It was from this particular case of it that the name *moment* arose, signifying that on which the power of a force in producing rotation depends. Thus, to close a door rotating on its hinge, by a push, it is much more effectual to apply the push near the handle than near the hinge. In pulling at a lever, the farther you are from the fulcrum the more power you have. Doubling the distance of the force from the fulcrum is as good as doubling the force itself—doubling either doubles the effect—doubling both quadruples it: hence distance and force enter equally into the effect—that is, the *moment* of the force is *proportional* to the product of force and distance, FR, and may be defined as *equal* to it.

The distance called R here is always the *shortest* distance from the point or axis of rotation to the line of action of the force—that is, it is the length of the perpendicular drawn between these two lines, or let fall from the fixed point upon the line of action. Now, the area of a triangle is half the base multiplied by the perpendicular height; hence, if the force be taken as the base of a triangle, and the point of rotation as the vertex, the area of the triangle so formed will be half the moment of the force about that point. Or, in symbols: the moment of the force AB about the point O is $AB \times ON$, where ON is the distance R, being the perpendicular let fall from O upon the line AB, produced if necessary (fig. 13); but $AB \times ON$ also equals twice the area of the triangle OAB; hence, twice the area of OAB represents geometrically the moment spoken of. The posi-

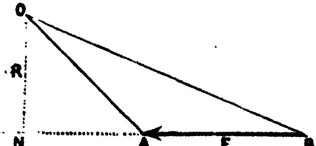


Fig. 13.

tion of AB *in the line* is evidently of no consequence, as all triangles of equal heights and bases have the same area (cf. sect. 51).

But to express a moment completely, we must also notice the *direction* of its rotative tendency. In the figure it happens to be like the hands of a watch, a direction it is convenient to call, with Professor Clifford, 'clockwise.' If AB were reversed, or if O had been on the other side of it, the direction of rotation would be also reversed, or 'counter-clockwise.' This last direction—namely, that opposite to the hands of a clock, it is customary to call positive,—the clockwise rotation being therefore negative. So in the above figure the moment is equal to 2.OBA; the *order* of quoting the angles being attended to.

Force Perpendicular to Path of Motion.

58. But there is another aspect of the subject. When a body (say a wheel) rotates round an axis, every point of it is describing a circle; and so, even when its motion is uniform, and not accelerated in the ordinary sense, still a force must act on each of its particles to compel them to move in the circle contrary to the first law of motion. This force is supplied by the strength of its material, and is often neglected; it is, however, very important. It may happen that the material of a wheel is not strong enough to exert the force required when the rotation is very rapid, and in that case the particles will cease to move in their circles, but will begin to move in straight lines: in other words, the wheel will fly to pieces. If a body revolve about a centre outside itself, the needful force must be supplied to it by a link or cord, or by some other constraining mechanism (a groove in the case of a solitaire marble running round its board).

In this, as in every case, the acceleration is proportional to the force, and a constant force produces a uniform acceleration (sect. 45); but the acceleration, or at least a component of it, is here perpendicular to the direction of motion

(see sect. 15), and this is a most important case. We will now proceed to investigate it further.

(III.) ON THE DIRECTION OF MOTION AS AFFECTED BY FORCE; OR, FORCE AND CURVILINEAR MOTION.

(Dynamics of a particle continued.) *(Centrifugal force.)*

59. The velocity of a particle of matter may be changed both in magnitude and direction by the action of force. Hitherto we have dealt only with change of magnitude; let us now proceed to change of direction; and consider a case where a force produces *only* curvature in the path of a particle without otherwise affecting the velocity.

Imagine a particle of matter moving round and round a circle with constant speed. Although there is no acceleration in the direction of its motion, yet, nevertheless, a force must, necessarily act continually in order that the *circular* motion may continue. The velocity is uniform indeed in speed, but its direction is constantly changing. But, by the first law of motion, a particle of matter will move always in the same direction—that is, in a straight line—unless it is acted on by force: hence, force is necessary to change the direction.

If the particle were at the point O (fig. 14), and the force were to cease to act, it would continue to move in the straight line OT, touching the circle at O. In order to go round the circle, it must then fall from this line normally, that is, toward the centre of the circle; it thus arrives at the point P, and now it is going along PT′; but it falls a little towards the centre again and so reaches the point Q, and so on. A force then must constantly act drawing the particle towards the centre

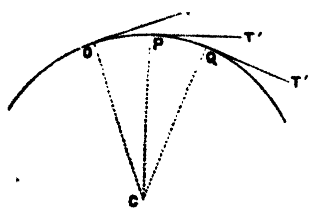


Fig. 14.

of the circle; and this force is called therefore the *centripetal* force. It is constant in magnitude, but continually changing in direction, being always at right angles to the direction of motion of the particle. And because it *is* at right angles to this direction, it can produce no acceleration *in* it. Whirl a stone round by a string: the tension in the string is this centripetal force, and you will find it greater as the stone is larger, and also as you whirl it quicker. The tension in the string, however, is really a stress (sects. 3 and 52), and has two aspects, one the *action* of the hand or central body on the revolving particle, which is the centripetal force proper; the other the *reaction* of the revolving particle on the central body, which is the force felt by the hand, and goes by the name of *centrifugal* force. Of course the two are equal. The essential thing, however, is the stress, and which component we speak of matters little: but, as we are at present concerned more with the action on the particle than with the reaction on the centre, it will be convenient to attend more to the centripetal force than to the other.

Value of Centripetal Force and Centripetal Acceleration.

60. Now, to find the magnitude of this force, we must regard the motion of the particle as compounded of two—one a uniform velocity along the tangent to the circle; the other a uniform acceleration along the radius, produced by the uniform centripetal force F, according to the law

$$a = \frac{F}{m},$$

a being the centripetal acceleration.

We have then a case of composition of motions very like that discussed in sect. 29, where a uniform rectilinear motion was compounded with a uniform acceleration in a constant

direction—that is, always parallel to itself; and where the path of the resultant motion was found to be a parabola. But, in the present case, we have to compound a uniform motion with a uniform acceleration at right angles to the path of motion at each instant, in fact along the radius of the circle, and by no means parallel to itself.

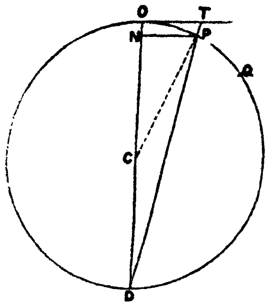


Fig. 15.

Drawing a figure similar in principle to that of sect. 29 (fig. 9, which see), let OP be the very minute portion of the circular path described in an infinitesimal portion of time t with the constant velocity v, so that

$$OP = vt;$$

and complete the figure as shown in fig. 15, letting fall PN perpendicularly to the diameter of the circle OD.

Then OP is the diagonal of an infinitely small parallelogram* with sides OT and ON; wherefore the motion along OP may be regarded as compounded of two motions—one with the constant velocity v along OT, which the particle would have if left to itself; the other, due to a constant pull of the centre C, and therefore uniformly accelerated, along ON, which is the distance travelled in that direction in the above small time t; wherefore

$$ON = \tfrac{1}{2}\,at^2.$$

It only remains to determine, from the geometry of the figure, the relation between ON and OP, in order to find

* The quadrilateral ONPT is not really a parallelogram, but it is more nearly one the smaller it is—that is, the nearer P is taken to O; and it is accurately one in the limit when it is infinitely small—that is, when P is the next consecutive point to O, which is supposed to be the case; for of course OPQ, &c. are really *consecutive* points of the circle, only they have to be spread out in the diagram. In the limit also OP and OT are equal, and hence OT is also equal to vt.

the value of the centripetal acceleration for a point moving with given velocity in a circle of given size.

The angle OPD, being an angle in a semicircle, is a right angle (Euc. III. 31), and so is the angle at N; moreover, the angle at O is common to the two triangles ONP and OPD; wherefore these triangles are similar (that is, one is like the other magnified), and their corresponding sides are therefore proportional; so

$$ON : OP :: OP : OD;$$

or in symbols, if r is the radius, CP, of the circle,

$$\tfrac{1}{2} at^2 : vt :: vt : 2r;$$

$$\text{or, } a : v = v : r,$$

whence $v^2 = ra$, or v is a 'mean proportional' or 'geometric mean' between a and r.

The value of the centripetal acceleration is then

$$a = \frac{v^2}{r}; \quad \text{or (writing } v = \omega r\text{)} \quad a = \omega^2 r = v\omega.$$

The centripetal *force* is of course simply m times this, m being the mass of the revolving particle of matter, or

$$F = \frac{mv^2}{r} = m\omega^2 r = mv\omega;$$

or it is proportional to the mass of the particle, the square of its velocity, and the curvature (see sect. 13) of its path: in other words, it is proportional not only to the momentum of the particle, but also to the rate at which its direction of motion revolves—that is, to its angular velocity.

Read again sect. 15 carefully, and also the examples on circular motion in Examples III., especially No. (3).

61. As an example take a stone weighing 5 lb., attach it to a string 3 feet long, and then whirl it round twice a second.

The length of one circumference being $2 \times \pi \times 3 = 6\pi$ feet, its

velocity must be 12π feet per second; and the tension in the string, or the centripetal stress, must be

$$\frac{5 \times 144 \times \pi^2}{3} = \text{about } 2400 \text{ poundals}$$

(taking π^2 as equal to 10 instead of 9·87); equivalent to the weight of $\frac{2400}{32} = 75$ lb.

This stress might easily be sufficient to break the string, and one would then *correctly* say that the centrifugal force, exerted by the revolving mass on the string, broke it. This may be understood as an abbreviation for the following more expansive statement: The force required to continually deflect the mass from its natural rectilinear path, and cause it to move in the given circle at the given rate, was so great that the string was incompetent to exert it, but was torn asunder in the effort.

Take another example from astronomy, which, however, will be better appreciated after reading Chapter V. The moon revolves round the earth, in a path which is nearly a circle with the earth as centre, in a time of nearly 28 days. Hence it too is continually being deflected from its natural rectilinear path: the force which deflects it being its *weight* —that is, the earth's pull (or gravitative attraction). Call the mass of the moon m; then its weight must be mg' (see sect. 64), where g' is the intensity of terrestrial gravity at the distance of the moon.

The intensity of gravity at the moon's distance is much less than 32, its value near the surface of the earth, because it decreases in the same proportion as the square of the distance from the centre of the earth increases.

This force, mg', then, is the centripetal force which makes the moon describe its curved path, and hence it should equal

$$\frac{mv^2}{r}.$$

Now the radius (r) of the moon's orbit is about 240,000 miles, or about sixty times the earth's radius; and it goes

once round in 2,360,000 seconds, or about 27 days 8 hours: hence its velocity (v) is

$$\frac{2\pi \times 240{,}000 \times 1760 \times 3}{2{,}360{,}000} = 3374 \text{ feet per second.}$$

So $\frac{v^2}{r}$, the centripetal acceleration, is $\frac{3374 \times 2\pi}{2{,}360{,}000} = \cdot 00898.$

This is the value of the acceleration above denoted by g', and the centripetal *force* is

$$\cdot 00898m.$$

Now if this force be really due to gravity, and if gravity really diminishes with the square of the distance, then, the distance of the centre of the earth from the moon being sixty times as great as its distance from the surface of the earth (that is, the earth's radius), it would follow that g' at the distance of the moon should be the 3600th part of the value of g at the surface of the earth.

But the value of g is 32·2 (see next chapter), and the 3600th part of this is ·00894; so the weight of the moon should be $\cdot 00894m$; and this is as nearly equal to the necessary centripetal force, $\cdot 00898m$, as our rough data can be expected to give it.

This is the sort of calculation which Newton went through when he proved that the force required to keep the moon in her orbit was just the same as would be exerted by the gravitative pull of the earth; supposing that the force which pulls down stones and apples extended so far, and decreased regularly all the way with the square of the distance from the centre of the earth. Whence he concluded that this force *does* so extend, and is the actual force in operation.

EXAMPLES—X.

(1) Half a pound is whirled at the end of a string 18 inches long 3 times round per second. What is the tension in the string?

(2) If a string can stand a force of 1000 units without breaking, what is the greatest length of it which can be used to whirl a 5-lb. mass once round a second?

(3) What is the smallest length of the same string which can be used to whirl a 5-lb. mass with a velocity of 10 feet a second?

(4) A string just able to carry 40 lb. is used to whirl a ½-lb. weight in a horizontal circle 5 feet in diameter. At what speed will the string break?

(5) Find the tension in each spoke of a six-spoked flywheel, 8 feet in diameter and weighing 12 cwt., when making 200 revolutions per minute, assuming all its mass collected at its rim, and that by reason of cracks in the rim, the spokes have to bear the whole of the strain.

As a matter of fact the rim of a flywheel ought to stand its own strain, and the spokes need add but little to the strength.

(6) A stone weighing 6 oz. is attached to the end of a cord 3 feet long, the other end being fixed. If the cord breaks when the stone is whirled round at the rate of 10 turns per second, find the greatest weight it could have been able to support.

CHAPTER V.

ON FORCE AND MOTION—*Continued.*

THE FORCE OF GRAVITATION.

62. Before proceeding further, it will help our ideas to apply some of the general laws to a few special cases. The most universal force known is the force of gravitation, and it will be convenient to take illustrations from the action of this force; but we will, in the present stage, only consider it as a *uniform* action exerted by the earth, tending to pull every piece of matter down to the earth's surface with a force varying with the mass of the piece of matter, but with nothing else. This is practically true in all common cases, for though the force really varies inversely with the square of the distance from the centre of the earth, yet the variation for ordinary heights is very small. For there is scarcely any difference in the distance of the centre of the earth from the sea-level and from the top of a mountain—one is say 4000, and the other perhaps 4001 miles.

63. This force is what is known as *weight*; it is measured like every other force by the acceleration it can produce in unit mass, or, in other words, by the momentum it can generate in a second. To measure the force, and see how it depends on the nature of the attracted body, we will first take the *same* mass of *different* bodies, and compare the accelerations which gravity is able to produce in them. Thus take a pound (see sects. 34 and 36) of lead, of iron, of stone, of wood, and of cork, and drop them all at the same instant from a high tower; then if every disturbing cause were absent—that is, if they were subject to no other force

but that of gravitation—they would all be found to reach the ground at precisely the same instant, having all acquired the same velocities.

If, however, the experiment took place in air, they would be subject to disturbing causes, and the falling together would be only approximate. The wood and cork would be retarded by the air more than the others, partly from the same cause that enables us to winnow chaff from grain, and partly for a reason which may be rendered more obvious by dropping the different things under water. The falling of the wood and cork would be then not only retarded but reversed into a rise. The air has a floating power, only it is less than that of water. The air must therefore be removed, and the bodies dropped *in vacuo,* if the observation is to be precise and extended to all sorts of bodies such as cotton-wool or feathers; the experiment is then often called the guinea and feather experiment, and for a description of it you may refer to Ganot or Deschanel.

The above experiment, if carried out accurately, would prove that *the pull of gravity has nothing to do with the material or nature of the substances*, for all equal masses are equally accelerated whatever the material; and, since the masses *are* equal, this means that they are all pulled with equal force (sect. 45).

64. Next take unequal masses (either of the same material or not), say a swan-shot and a cannon-ball, and drop them from a height at the same instant. They will both reach the ground at the same time—that is, they each receive the same acceleration. This experiment was carried out by Galileo from the Tower of Pisa. It shows that *the earth's pull on a body is directly proportional to its mass.* For since force is equal to the product of mass and acceleration (sect. 45), and since the acceleration is found experimentally to be the same for all masses, it follows that the force is necessarily proportional to the mass.

If we denote by g the acceleration produced by gravity —that is, the velocity gained per second by a freely

falling body—the force pulling it down is measured by its mass multiplied by g, and this force is termed its *weight*; so

$$W = mg;$$

or, the weight of a body is g times its mass; in other words, g = weight ÷ mass, that is, weight per unit mass. Hence g is often called the *intensity* of gravity.

This is simply a special case of the general relation

$$F = ma,$$

weight being a particular case of force, and g being a particular case of acceleration. The acceleration due to gravity is the same for all material bodies, and if one thing is ever observed to fall more slowly than another, it is because of the disturbing effect of the air. In a vacuum all things fall at the same rate if they start fair; and this experimental fact, combined with the second law of motion, proves that weight is proportional to mass.

The ratio of weight to mass is often called the intensity of gravity, and denoted by g; and a knowledge of its value at any place enables us to translate gravitational units of force there into invariable absolute units. In this country, g is found by experiment (see next section) to be about 32·2 F.P.S. units, or about 981 C.G.S. units; so that the weight of a pound is about 32·2 poundals, and the weight of a gramme about 981 dynes. If the earth were a stationary sphere, g would be constant all over its surface, and it would only vary on ascending above or descending below the surface; as the inverse square of the distance from the centre as you ascend, as the direct simple distance from the centre as you descend (supposing the density uniform). But inasmuch as the earth is a rotating spheroid, the intensity of gravity is different at different parts of the surface, as shown in the following table, whose figures are calculated from the

known shape of the earth, which, again, was calculated by Newton from its period of rotation.

INTENSITY OF GRAVITY AT DIFFERENT LATITUDES.

	Latitude.	g in C.G.S. units.	g in F.P.S. units.
Equator................	0°	978·10	32·090
	45°	980·61	32·173
Paris....................	48°·50′	980·94	32·184
Greenwich...	51·29	981·17	32·191
Berlin..................	52·30	981·25	32·194
Liverpool..............	53·29	981·34	32·197
Edinburgh	55·57	981·64	32·207
North Pole...........	90°	983·11	32·255

Thus, whereas a stone falls 16·045 feet in the first second at the equator, it falls 16·099 feet at Liverpool, and 16·128 feet at either pole.

The weight of one pound is therefore about 32·2 absolute F.P.S. units of force or poundals. (Read sect. 36 again.)

Falling Bodies.

65. To express all the laws of falling bodies, we have simply to apply all the kinematics we know about bodies moving with constant acceleration.

Thus a stone let drop is found to fall about 16 feet in one second (more accurately 16·09), so that 16 is its average velocity during that second; but the average velocity is half the final velocity; hence the velocity acquired in one second, or the acceleration, is 32 (more accurately 32·18), and this is the value of g. (Since 16·09 feet equal 490·5 centimetres nearly, the value of g is 981 in centimetres-per-second per second; and this therefore is the weight of a gramme in dynes.) A more accurate method of determining

g is to let a body fall again and again, and take the time of a long succession of falls. This can be practically carried out with a pendulum (see sect. 78); but for many purposes an approximate value of g in round numbers is sufficient, and the number 32 happens to be very convenient, because both its half and its double have an easy square root. Hence the following handy rules:

The velocity 32 is gained in every second of the fall, so the velocity gained in t seconds is $32t$ *(feet per second).*

The distance travelled, being proportional to the square of the time, is $16t^2$ *feet* ($h = \frac{1}{2}gt^2$).

The velocity gained while falling from a height of h feet from rest is (by equation $v^2 = 2gs$*), in feet per second,*

$$8\sqrt{h};$$

8 being the square root of 64 or $2g$.

The time taken to fall from rest at a height of h feet is, in seconds,

$$\tfrac{1}{4}\sqrt{h},$$

which follows at once from the equation $s = \frac{1}{2}gt^2$.

These expressions, $32t$, $16t^2$, $8\sqrt{h}$, and $\frac{1}{4}\sqrt{h}$, only apply when everything is expressed in feet and seconds, and when the falling body starts from rest; but given these conditions, they are very useful for rapid mental estimates, especially the last two. For instance, the time taken to drop 144 feet is $\frac{1}{4}\sqrt{144}$, or 3 seconds, and the velocity acquired is $8\sqrt{144}$, or 96 feet a second. To drop 400 feet, 5 seconds are needed, and the speed attained is 160.

66. Modes of diluting the Intensity of Gravity.—The acceleration is equal to g for all bodies only on condition that they fall *freely*—that is, that the weight of each has only its own mass to move and nothing else; for then $a = F/m$, but $F = mg$, so $a = g$.

If, however, by any arrangement, we make a weight move

another mass as well as its own, the acceleration must be less.

Thus suppose we tie a falling weight P (say 6 lb.) to a mass Q of 18 lb. resting on a *smooth* flat table, as in fig. 16; then the force causing the motion is the weight of the 6 lb.—that is, $6g$—but the total mass moved is $18+6=24$ lb.; hence the acceleration is

$$\frac{6g}{24}=\tfrac{1}{4}g=8$$

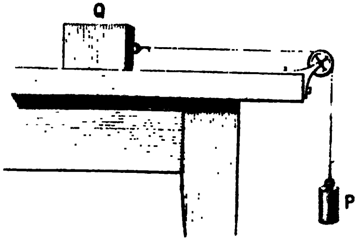


Fig. 16.

feet-per-second per second. Hence in the first second the combination would move 4 feet, and in t seconds $4t^2$ feet, while the velocity acquired in t seconds would be $8t$ feet per second.

In a similar way, we can find the acceleration if two weights are connected by a string passed over a frictionless pulley, without inertia, as in fig. 17.

Let the masses Q and P be 7 and 9 lb., their weights will be $7g$ and $9g$ respectively, and the effective force will be the difference, that is, $2g$. The mass moved is 16; hence the acceleration is

$$\frac{2g}{16}=\tfrac{1}{8}g=4 \text{ F.P.S. units.}$$

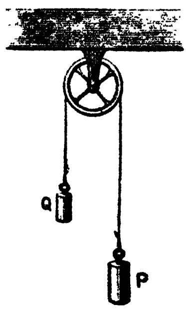


Fig. 17.

This arrangement of two unequal weights over a pulley is called 'Atwood's machine,' for determining the acceleration produced by gravity, and for experimenting on the laws of uniform acceleration.

The advantage gained by experimenting with it instead of with freely falling bodies is owing to the fact that the latter fall too quickly to be conveniently observed. Any acceleration whatever less than g can be obtained by the use of this simple machine. Gravity is as it were *diluted* (that is,

its intensity has to be multiplied by a proper fraction to give its accelerative effect), but the laws of falling remain the same. Another mode of diluting gravity, employed by Galileo, is to roll round bodies down an inclined plane.

67. Mode of measuring the Intensity of Gravity.—To use the machine for measuring g, we put on the string two nearly equal weights, masses P and Q; the effective force is then the difference of their weights, $Pg - Qg$; the mass moved is $P + Q$; hence the acceleration is (by direct application of Newton's second law, in the form $a = F/m$, the ratio of the moving force to the mass moved)

$$\frac{Pg - Qg}{P+Q}, \text{ or } \frac{P-Q}{P+Q}g.$$

If this acceleration (call it a) is observed, g is easily calculated. To obtain a experimentally, you may observe the distance s fallen in time t, and then apply the formula $s = \frac{1}{2}at^2$.

For instance, let P be 21 oz. and Q be 23 oz., then the acceleration is, as calculated dynamically,

$$\frac{2}{44}g, \text{ or } \frac{g}{22}.$$

Let the weights move for six seconds, repeating the experiment several times to get it fairly accurate, and observe that the heavier weight has fallen (and the lighter weight risen) a distance of 26 feet; then say 26 feet $= \frac{1}{2}a \times (6 \text{ sec.})^2$, whence $a = \frac{13 \text{ feet}}{9 \text{ (sec.)}^2}$, or the acceleration observed is $\frac{13}{9}$ F.P.S. units; but it has just been calculated as also $\frac{g}{22}$, hence $g = 31\frac{7}{9}$.

An actual experiment with Atwood's machine would be hardly likely to give g so nearly correct as this. There are other methods of finding the value of g which are much better practically, though not so theoretically simple. The

most accurate method consists in observing the time of oscillation of a pendulum of measured length, which may be considered as a heavy body constrained to fall along a circular path, and having its fall repeated automatically again and again, so that the time of fall may be accurately measured—namely, one quarter of the period of a complete swing to and fro (see sect. 78).

68. It is easy to understand how experiments may be made with Atwood's machine on the laws of uniform acceleration. Thus, to take the case when the weights are in the ratio of 21 to 23, and the acceleration therefore $\frac{13}{9}$, we should find that the distances travelled in 1, 2, 3, 4, 5, 6 seconds respectively were, in fact,

$$\frac{13}{18}, \frac{4\times 13}{18}, \frac{9\times 13}{18}, \frac{16\times 13}{18}, \frac{25\times 13}{18}; \frac{36\times 13}{18}$$

—that is, always half the acceleration multiplied by the square of the time ($\frac{1}{2}at^2$).

The distances travelled *during* each second would follow another law. They are easily obtained from the preceding numbers, for if we subtract the distance travelled in three seconds from the distance travelled in four, we obtain the distance travelled *during the fourth second*—namely,

$$\frac{16\times 13}{18}-\frac{9\times 13}{18}=\frac{7\times 13}{18};$$

and similarly, we get for the distance travelled in the first, second, third, fourth, and fifth seconds respectively,

$$\frac{13}{18}, \frac{3\times 13}{18}, \frac{5\times 13}{18}, \frac{7\times 13}{18}, \frac{9\times 13}{18};$$

a series ascending by the odd numbers; the distance travelled in the nth second being that travelled in the first second multiplied by the nth odd number, $\frac{1}{2}a(2n-1)$.

69. All this may be readily remembered by observing its analogy with a simple geometrical diagram, as in sect. 23.

Draw any right-angled triangle, OPC (upside down does best for falling bodies); divide its base, OP, into any

number of equal parts, and draw a vertical line at each division. You will thus cut up your triangle into trapeziums, of which the left-hand one degenerates into a triangle; and it is plain to simple inspection that, whatever be the area of this small triangle, the trapezium next to it has three times that area, the next five times, the next seven times, and so on, as may be seen from the dotted lines drawn in fig. 18.

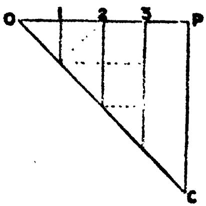


Fig. 18.

Hence, if the first area represent the distance travelled by a uniformly accelerated body in the first second, the second area will represent that described in the second second; and the sum of the two figures will be the space described in the two seconds together, and so on.

Moreover, the whole area of the triangle will represent the space travelled in the whole time, as measured by a number of seconds equal to the number of segments of the base. Thus, in the above figure, the whole area is the space described in four seconds. (Read sect. 23 again.)

The vertical height of the figure being nothing at its left-hand point, corresponds with the fact that the falling body starts from rest—that is, is *dropped*. But if the body is *thrown* either down or up with an initial velocity, this velocity must be represented by a line drawn at the left-hand point, either down or up, and the figure becomes as in fig. 19 or as in fig. 20, where OA represents a velocity downwards, and OA′ a velocity upwards.

In the first case the velocity continually increases, until in four seconds it becomes equal to PC. In the second case it at first decreases, becoming zero at the point E two seconds after starting, and then increases downwards until it becomes P′C′.

This second case exactly corresponds with that of a ball thrown up in a vacuum against gravity. In both cases the

whole area of the figure represents the whole space travelled. In the second case we see that the area OA′E is the space or height the ball rose through, and EP′C′ the height it afterwards fell through. The ball was at its highest

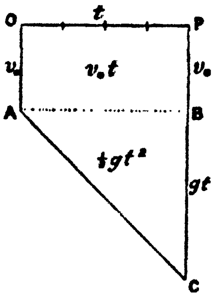


Fig. 19.

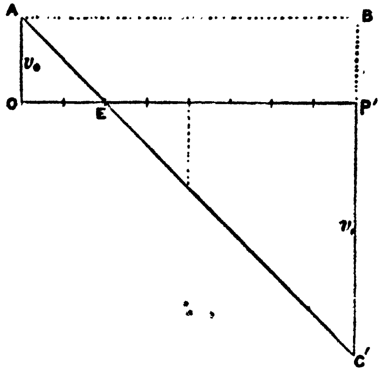


Fig. 20.

point two seconds after being thrown up, having then no velocity.

In both cases the ball must have been thrown from the top of a tower or some other height, or it could not fall for so much as four seconds without striking the ground. The area OACP may represent the height of this tower, and OP the time taken to fall, in the first case—that is, when the ball was thrown downwards; but in the second case, when the ball was started upwards, the height of the tower is the difference of the areas EP′C′ and OA′E. OP′ is the whole time taken by the ball, first to rise a height above the tower equal to the area OA′E, and then to fall from this height to the ground. The lines AC and A′C′ are necessarily parallel, since the slope of each represents the rate of numerical gain of velocity, 32 feet-per-second per second. (It might, however, be anything less than this if Atwood's or some other 'diluting' machine were used.)

Supposing it is 32, and that OE is two seconds, and EP′ six; then, of course, the initial velocity OA′ must be 64, and P′C′ must be 192, feet per second. The area OA′E will be 64 units (its height being 64, and its base 2), and therefore the height the ball rises is 64 feet. The area EP′C′ is 576, and so the height of the tower is 512 feet. In the diagrams, the time represented by OP′ in the second diagram is greater than the time, OP, in the first, by twice OE; and the initial velocity OA is numerically equal to OA′; hence also the final velocity PC is equal to P′C′, and the area OACP represents 512 linear feet.

In each of these figures (neglecting dashes), OP is the line of time; OABP represents the space described due to the initial velocity alone; and ABC the space described due to gravity. Also BC represents the gain of velocity at; and PC the actual final velocity.

Refer to sect. 23 for some more statements concerning these diagrams and practise drawing diagrams for all kinds of cases of rectilinear motion. Thus, draw diagrams for the motion of a railway train, which gets up speed, goes uniformly, slackens, stops, and goes on again, several times, and then comes back; for the motion of an india-rubber ball thrown down to the ground and then bouncing; for the motion of the bob of a very long pendulum; for the motion of a tilt hammer, &c.; and remember in drawing these diagrams that *time* never retrogrades, and hence that no part of a diagram can be vertically under or over another part, but the drawing must progress continually forwards. Journeys back are represented by areas below the line of time instead of above. The curve for a bouncing ball is practically a discontinuous set of parallel straight lines, because at the moment of bounce the velocity is suddenly reversed.

70. To actually experiment on the velocity acquired by the falling weights in Atwood's machine, we must remember the definition of variable velocity at any instant (given in Chapter I., end of sect. 9)—namely, the distance the body would go in the next second if at that instant the accelera-

tion ceased. Now, *the cause of the acceleration in this machine is the force* $(P-Q)g$. *If this force were suddenly removed*—that is, if P and Q were suddenly made equal, there would be no further acceleration, and the masses *would continue to move uniformly forward* with the velocities they had already acquired, until they were checked either by striking something or by friction.

This sudden removal of the inequality in the two weights is practically accomplished by making the extra weight by which P exceeds Q (2 ounces in the experiment of sect. 67), a loose metal bar too big to pass through a certain fixed ring placed in the path of P. When P passes through this ring the bar is removed; P and Q become equal, and move a distance in the next second which is numerically equal to the velocity they had acquired at the instant the bar was taken off. For a fuller description of Atwood's machine, and for many details of its actual construction, you may refer to Deschanel or Ganot.

71. Further Illustration of the Fundamental Equation.—This example of the two weights, one pulling up the other, illustrates the statement in sect. 54 that the second law of motion applies to other cases than those where the motion is perfectly free and unresisted; in fact, that it is quite general, if we always consider the force F as the *resultant* of all the forces acting on a body, and not simply that force which happens to be most obviously apparent to us.

Thus, go back to the mass of 18 lb. resting on a table, and pulled along by a weight of 6 lb. hanging over the edge by a string (fig. 16, sect. 66). The acceleration we saw ought to be 8, but suppose it was observed to be only 3, we should at once know that all the forces had not been taken into account. The table, perhaps, is rough, and retards the motion of the 18 lb. with a force sufficient to reduce its acceleration to 3, and the force of friction may from these data be calculated.

So again when a 56-lb. bucket is dragged up a well with a force of 1920 poundals (the weight of 60 lb.); if this were the only force acting, the acceleration of the bucket upward would be

$$\frac{1920}{56}=34\frac{2}{7}$$

F.P.S. units—that is, it would gain this velocity per second; but if the experiment be tried, the velocity actually gained per second will be found to be nothing like so much as this—it will be only about $2\frac{2}{7}$ units. The reason obviously is that there is another force left out of account, opposing the pull of the rope—namely, the pull of the earth, which is $56 \times 32 = 1792$ poundals; and the resultant force is the difference of these two, or 128. Hence the actual acceleration is

$$\frac{128}{56} \text{ or } 2\tfrac{2}{7}.$$

72. To take another very similar example, a cage of mass m, or say 1000 lb., is lowered by a rope down a coal-pit; what is the tension in the rope, at a time when the cage is gaining downward velocity at the rate a, or say 24, feet a second every second? Well, the resultant force must equal the mass acceleration, but this resultant force is the difference between the weight of the cage, mg, and the pull of the rope, T, hence

$$mg - \mathrm{T} = ma,$$

or

$$\mathrm{T} = m(g - a),$$

which is the tension required. Numerically, in the present illustration, the tension is 1000×8 poundals $= \frac{8000}{32}$ lb. weight $= \frac{1}{4}$ of the normal weight of the cage. The tension in the rope is always $\frac{g-a}{g}$ times the weight of the cage if a represents its downward acceleration. If the cage is being accelerated *upwards*, the tension in its rope is $\frac{g+a}{g}$ times its weight.

If the tension in the rope were *equal* to the weight of the cage, the cage would necessarily have a constant velocity; it might be moving either up or down, but there could be no acceleration (cf. sect. 133). Such a body is obeying the first law of motion; it is subject to no resultant or unbalanced force. A locomotive dragging a train at constant speed on a straight line, even uphill, is in the same case; the forces on it are balanced; and until the rails curve, or the steam is shut off, or the brakes are put on, the motion is perfectly uniform, the effective force is zero.

EXAMPLES—XI.

(1) What is the weight of 20 lb. at a place where a falling body travels 4 feet in the first second?

(2) At what height above the earth's surface could such a place be found?

(3) A curling weight is thrown on ice with a velocity 50 ft. per second. Supposing the force of friction to be $\frac{1}{10}$th of the weight, how soon will it stop?

(4) In an Atwood's machine a 40-gramme weight on one side is drawn up by a 50-gramme weight on the other, 2·18 metres in two seconds. What is the value of g in centimetres-per-second per second?

(5) In the preceding question find the tension in the cord in grammes weight, and in dynes.

(6) When a 3-lb. weight hanging over the edge of a smooth table drags a 45-lb. mass along it, find the acceleration and the tension in the string.

(7) Find also the acceleration if the coefficient of friction between the table and the weight is ·05.

(8) From a balloon which is ascending with a velocity of 80 feet a second a stone is dropped, and seen to strike the ground in 7 seconds. Find the height of the balloon at the moment of letting go the stone.

(9) A cage is hauled up from the bottom of a mine with an acceleration of 12 feet-per-second per second. After rising 96 feet a stone is dropped from it. How soon will the stone reach the bottom of the mine?

(10) If a body fell at a certain place 50 feet in the 3d second of its fall from rest, what would be the intensity of gravity at that place?

(11) A body weighing 30 lb. placed on a smooth horizontal table has a string attached which runs parallel to the table, passes over a smooth peg at the edge of the table, and has a weight of 2 lb. hanging at its end. Find the tension in the string, and the acceleration of the system.

(12) Two bodies of 17 and 15 lb. respectively are connected by a string which passes over a smooth pulley. Find the acceleration with which they will move, and the tension of the string.

(13) Determine the acceleration of an Atwood's machine if the masses at the ends of the thread are 40 and 50 grammes, and the pulley is equivalent to an additional mass of 10 grammes.

(14) In an Atwood's machine the weights on either side are 5 and 4 lb. respectively, and the wheel is equivalent to an extra

inertia of 3 lb. Find the distance descended by the heavier weight in 3 seconds, and the velocity acquired. Find also the tension in each portion of the light cord.

(15) If the weights are respectively $3\frac{1}{2}$ and $4\frac{1}{2}$ lb., and if the motion is 12 feet in 3 seconds from rest without friction, what inertia must the wheel be equivalent to?

(16) A mass of 20 lb. is being dragged off a table by a 4-lb. weight attached by a string which passes over a smooth pulley at the edge of the table. The friction between the weight and the table is $\frac{1}{15}$ of that weight. Determine how far the whole moves in 3 seconds from rest, and find the tension in the string.

(17) If a stone is dropped out of an ascending or descending balloon, how long a time will elapse before it is 16 feet below the balloon? If a stone and a light fleck of cotton-wool are released simultaneously and are in one second 30 feet apart, how rapid is the vertical motion of the balloon? If the stone and the wool are observed to meet and pass each other 3 seconds after their simultaneous release, what is the balloon doing?

Assume in each case that the wool remains practically stationary, suspended in air, and that the balloon's motion, whatever it is, is steady.

(18) An engine winds a three-ton cage up a coal-pit shaft at a uniform pace of 11 yards a second. What is the tension in the rope?

(19) Instead of a uniform *velocity*, the above cage is wound up with a uniform *acceleration* 6 feet-per-second per second. What is the tension in the rope?

(20) A monkey clings to a light flexible rope passed over a large fixed pulley without inertia or friction, and is balanced by a precisely equal weight at the other end of the rope. What happens if it now begins to climb the rope?

73. That aspect of the second law of motion which says that it makes no difference to the effect of a force on a body whether that body was in motion or not (sect. 49) is well illustrated by falling bodies (see sect. 22).

74. But the law is more strikingly illustrated when the direction of the initial velocity of a falling body (now called a *projectile*) is inclined at some angle to the *force* of gravity.

The path of a projectile is shown in fig. 9, sect. 29; the simplest case being where the initial velocity is at right angles to the force of gravity, or horizontal.

Thus a rifle-bullet, starting with an initial horizontal velocity u, retains this velocity unaltered, if we neglect friction against the air, and therefore in t seconds it travels a horizontal distance ut; but its vertical velocity, which at first was zero, continually increases, and in t seconds is gt; the vertical space fallen through being $\frac{1}{2}gt^2$, or just the same as if the gravity had acted upon the body at rest. The whole circumstances of the motion of such a projectile have therefore been already worked out in sect. 29; which see, and read again.

If the rifle is fired horizontally from the top of a cliff of given height, say 144 feet, it is easy to find how far the bullet will go before striking level ground, its initial velocity being known. Let the initial horizontal velocity be 1200 feet per second. We must first find t, the time the bullet takes to fall from the top of the cliff to the ground, from the equation $144 = 16t^2$ (for it takes just the same time as if it had no horizontal velocity. Law II., sect. 49); this gives $t^2 = 9$, or $t = 3$. It goes therefore three seconds before striking the ground, so evidently the horizontal distance it travels is $3 \times 1200 = 3600$ feet.

And generally if h be the height of the cliff, and u the initial horizontal velocity of the bullet, its *range*, or horizontal distance, is $\frac{1}{4}u\sqrt{h}$.

75. The composition of these two motions, a uniform horizontal velocity with a uniform vertical acceleration, is well illustrated by Morin's machine, for a description of which see Deschanel or Ganot.

It consists of a long drum or cylinder, capable of rotating by clockwork about a vertical axis. Down one side a weight can fall between guides, and can, by means of a pencil, mark a line on the drum as it falls. If the drum is stationary, the line drawn is, of course, straight and vertical; but if the drum rotates, it is spread out into a curve. This curve, when un-

wrapped from the drum, is precisely the same as that which is described by a projectile shot out horizontally *in vacuo* with a velocity equal to that imparted to the surface of the drum by its clockwork.

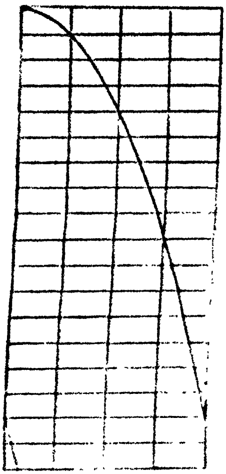


Fig. 21.

The drum is usually covered with paper, ruled into squares or oblongs, which can be detached and unrolled. The line traced on it may then present the appearance shown in fig. 21. In successive seconds the horizontal distances are as 1, 2, 3, 4, 5,...... the vertical as 1, 4, 9, 16, 25, and so on. *A curve with this property is called* a parabola. It is the path of a projectile in a vacuum (compare sect. 29).

76. *The simplest method of dealing* with projectiles is to resolve their initial velocity of projection into a horizontal and a vertical component, and to treat them separately. The horizontal motion is not subjected to any force except the resistance of the air; the vertical motion is subject to gravity as well. It is a complicated matter to take the resistance of the air into account; especially since, if the projectile is spinning, the air resistance directly alters its path as well as its speed. Suffice it to say that the simple parabola could only be really attained in a vacuum, and that the path of a simply thrown cricket-ball is an unsymmetrical curve with its descending portion shorter and steeper than its ascending portion. In the case of a golf-ball, the spinning motion so complicates matters that it may travel straight for some distance, and then actually rise upon the air resistance and drop fairly dead. Or, if cut sideways, like a racquet-ball, it may describe a curved path not by any means in one plane. We shall limit ourselves to the case of a body shot from level ground in a vacuum, with a certain initial velocity inclined at a certain angle to the horizon. Given these data, it is easy, by constructing a parallelogram, or otherwise, to determine its horizontal component, which we will call u, and its vertical component, v. Then the following statements should be proved, and examples on them should be worked:

The time, T, taken for a projectile to reach its highest point is given by........................$v=gT$.

Its whole time of flight is..............................$2T$.

Its *range*, or horizontal distance reached, is........$2uT = \frac{2uv}{g}$.

Its maximum elevation is ....$h = \frac{v^2}{2g}$.

Its vertical component of velocity, at any instant t, is ....$v - gt$.

Its horizontal component of velocity is always....u.

Its actual velocity at any instant, t, is.............$\sqrt{(u^2 + (v - gt)^2)}$.

Its actual velocity at any elevation, y, is..........$\sqrt{(u^2 + v^2 - 2gy)}$.

Its position at any instant is given by the two components, horizontal $x = ut$, vertical $y = vt - \frac{1}{2}gt^2$; from which, eliminating t, we get its trajectory or path—namely, the curve $u^2y = uvx - \frac{1}{2}gx^2$, which is a parabola with its axis vertical.

EXAMPLES—XII.

Projectiles in Vacuo.

(1) A cannon-ball is fired horizontally from a hill 900 feet high on the coast. Find the time which elapses before it strikes the sea, neglecting the resistance of the atmosphere.

(2) If the velocity of projection in the preceding question were 1320 feet per second, find the horizontal distance travelled.

(3) A string, 2 feet long, able to sustain a weight of 104 lb. without breaking, is attached to a stone weighing 4 lb. and whirled in a vertical plane round a fixed centre 6 feet above the ground till it breaks. What happens to the stone?

(4) A stone is dropped from the car of a balloon sailing along horizontally at the rate of 40 feet a second, 100 feet above the ground; find when, where, and with what velocity the stone will strike the ground.

(5) A bullet is fired with a velocity of which the horizontal and vertical components are 80 and 120 feet per second respectively. Find its range and greatest height.

(6) Two bullets are shot horizontally over a lake from the top of a tower 144 feet above the water; one of them with a velocity 300 feet per second, the other with 600. At what distances from the tower, and how long after leaving it, do the bullets strike the water?

(7) A stone is thrown obliquely with a velocity whose initial vertical and horizontal components are 160 and 96 feet per second respectively. Find its time of flight, and the greatest height it attains.

(8) A cricket-ball is driven so that it reaches a height of 75 feet and strikes the ground, supposed level, 300 feet away from the starting-point. Determine the magnitude and direction of its initial velocity.

(9) A ball is thrown with a velocity the horizontal and vertical components of which are 80 and 40 feet per second respectively. Find its range, the greatest height reached by it, and the time which elapses before it strikes the ground again.

(10) The horizontal component velocity of a cannon-ball is 800 feet per second. What was its vertical component at the moment of projection if it travelled 2000 yards range before returning to the level of the muzzle.

(11) A cannon-ball is projected at an angle of 30° to the horizon with a velocity of 2000 feet per second. Calculate its range, and the greatest height to which it will attain.

(12) What velocity must be given to a golf-ball to enable it to just clear the top of an obstacle 12 feet high and 100 yards distant, if the ball is struck upwards at an angle of 45°?

(13) Six bullets are simultaneously projected from the top of a 100-feet tower over level ground; one being sent horizontally, another at 30°, another at 45° elevation, another at 60° elevation, another at 30° depression, while the last is dropped. Find when and where they each strike the ground, and their maximum elevation; the initial velocity of each being the same—namely, 1200 feet per second.

Curvilinear Motion and Rotation.

77. We have already illustrated one other case of the *curvilinear motion of a particle* (sect. 59) produced by the force of gravity, namely that of the moon, supposing it to be a particle and to move in a circle (see sect. 61, and read it again). The whole subject of the motion of the planets in their orbits comes properly in here, but it is hardly profitable to attempt it at the present stage.

78. The subject of the *rotation of a rigid body* (sect. 54) under the action of gravity may be illustrated by fixing a point of a rigid body, and then letting gravity act on it. We thus get a very important set of physical laws known as those of the *pendulum*; for a 'pendulum' is simply a rigid body, with either a point or a line in it fixed somehow relatively to the earth, and then the body displaced from its position of equilibrium, and left to swing under the action of gravity. The motion is periodic, and the rate of oscillation depends only on the length of the pendulum (or its virtual length, as will be explained directly) and the intensity of gravity. The time of a complete swing to and fro is obtained by multiplying twice the ratio of the circumference of a circle to its diameter by the square root of the ratio of the length of the pendulum to the intensity of gravity—that is, in symbols,

$$t = 2\pi \sqrt{\frac{l}{g}}.$$

Assuming this (which will be practically proved in the next section), one sees that, by measuring t and l, the value of g can be ascertained ($g = 4\pi^2 l/t^2$); and this is the most accurate means of determining g. A pendulum whose length is obvious is called a simple pendulum; in other cases some pains must be bestowed on understanding and measuring the virtual or equivalent 'length,' a matter which is explained in sect. 80.

The practical use of a pendulum as a timekeeper depends on the time of an oscillation being almost invariable—that is, on its motion being *on the average* very uniform; hence it is very largely used as a timekeeper, all the rest of the clock being, firstly, an apparatus to keep the pendulum going notwithstanding friction, and, secondly, an apparatus to record (like a gas-meter) how many times the pendulum has oscillated.

Pendulums.

79. *Conical Pendulum and Governor Balls.*—Let AB (fig. 22) be a vertical axis of rotation, and P a massive ball at the end of an arm AP, capable of rotation about this vertical axis and pivoted at A; then it is well known that AP will fly out from the vertical more and more as it revolves faster and faster. Let it be revolving with a constant angular velocity ω, and let it perform every revolution in T seconds, so that $2\pi = \omega T$.

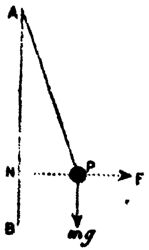


Fig. 22.

The centripetal force which must be acting on P in the direction PN to keep it moving in the circle (sect. 60) is $m\omega^2 r$, where r is the radius PN of the circle in which P moves; and if the rotation were to cease, this is the force which must be applied in the opposite direction PF, in order to keep the ball in its position without letting it fall back to the axis AB.

Hence in the diagram (fig. 22), we may regard P as stationary and in equilibrium under the action of three forces—the force $F = m\omega^2 r$, its weight $W = mg$, and the tension in its supporting arm. The triangle APN has its sides parallel to these forces, and hence represents them; so, calling the vertical distance AN, h, we have

$$m\omega^2 r : mg :: r : h;$$

or

$$h = \frac{g}{\omega^2};$$

that is, the vertical distance of the governor ball below the pivot A is inversely proportional to the square of the angular velocity of rotation.

The time of one revolution is $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{h}{g}}$; and such an arrangement is sometimes used as a measurer of time, when it is called a 'conical pendulum,' because the arm AP traces out a cone.

If the radius of the circle in which P moves is very small, the height h is practically equal to the length of the pendulum, AP, which we will call l. Moreover, if you try swinging a weight at the end of a string, you will find that the time of a complete *small* motion is the same whether the pendulum simply oscillates in a nearly straight line or whether it revolves in a horizontal circle or in any other elliptical curve; that is, the time of an

oscillation (to and fro) of a simple pendulum equals the time of rotation of a conical one, provided the motion of both is small; and each period is very approximately

$$2\pi\sqrt{\frac{l}{g}}.$$

By a *simple* pendulum is meant one about whose length there can be no ambiguity. It is a heavy *particle*, swinging at the end of a perfectly light cord attached to a fixed point.

EXAMPLES—XIII.

(1) Find the time of beat (half an oscillation is called a beat) of a simple pendulum 39 inches long at the equator, where $g=32\cdot09$.

(2) Where the length of a pendulum which beats seconds (called the seconds pendulum) is 39 inches, find the value of g.

(3) If gravity is $\frac{1}{190}$ greater at the north pole than at the equator, how many seconds a day will a seconds pendulum at the north pole lose when taken to the equator?

(4) How many seconds a day will a clock lose whose pendulum (intended to beat seconds) is a metre long, at a place where the intensity of gravity is 981 dynes per gramme? Find the length of the seconds pendulum at the same place.

(5) If a simple pendulum 39 inches long beats seconds, what should be the length of one which shall beat 40 times in one minute?

(6) Calculate the number of beats per day made by a simple pendulum 40 inches long at a place where the length of the seconds pendulum is 39 inches.

(7) If a pendulum 39 inches long is gaining a minute a day, how much should it be lengthened to keep correct time?

(8) What is the length of the seconds pendulum at the latitude of Greenwich? (See sect. 64.)

(9) What is the value of g at a place where a simple pendulum $2\frac{7}{16}$ inches long makes two complete oscillations a second?

(10) Show that the time of beat (in seconds) of a simple pendulum is, approximately, $0\cdot16\sqrt{(\text{length in inches})}$. Find the value of g for which this formula is accurate.

(11) Show that, if $g=981$ dynes per gramme, the time of a beat (in seconds) $=1\cdot00303\sqrt{(\text{length in metres})}$. Find the number of beats per minute of pendulums whose lengths are respectively 9 cm., 25 cm., and 16 metres.

(12) Find the number of beats per minute of a pendulum 3 feet long, using the formula of Ex. (10).

(13) If a weight be attached to a string 4 feet long, and is then caused to describe a horizontal circle, so that the string is inclined at 60° to the vertical, find its angular velocity, its actual velocity, and the time of one revolution.

80. *Compound Pendulum.*—The time of oscillation of a compound pendulum, that is, of a rigid body of any size pivoted on an axis through O, and swinging slightly under gravity, may now be calculated.

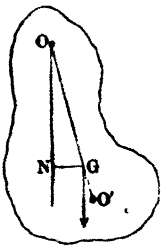


Fig. 28.

Let G be the centre of gravity of the mass, and call the distance OG, a; the small angle of displacement from the vertical, NOG, call θ; and the distance NG, call x; the latter is practically equal to $a\theta$, the arc of a circle with centre O.

Then, if m be the mass of the whole body, the force restoring the body to its position of equilibrium is mg acting at G, so that its moment about O is mgx; and the angular acceleration produced by this is (see sect. 56)

$$a = \frac{mgx}{I} = \frac{mga\theta}{I},$$

where I is the moment of inertia of the body about the point O. For the particular case of a simple pendulum, when the whole mass is concentrated into a particle at G, and when $a = l$ and $I = ml^2$, this equation becomes

$$a = \frac{mgl\theta}{ml^2} = \frac{g\theta}{l}.$$

Now we can choose a simple pendulum of such length that its angular acceleration at every instant, and therefore its whole motion, is the same as for the compound pendulum. Let L be the length of such an equivalent simple pendulum, then the equation

$$\frac{mga\theta}{I} = \frac{g\theta}{L}$$

is satisfied; and the length of the equivalent simple pendulum (sometimes called the 'length' of the compound pendulum itself) is

$$L = \frac{I}{ma}.$$

But the time of a small oscillation of this simple pendulum is

$$2\pi\sqrt{\frac{L}{g}};$$

therefore the time of a small oscillation of the compound pendulum is

$$T=2\pi\sqrt{\frac{I}{mga}},$$

where I stands, as already stated, for its moment of inertia Σmr^2 about the centre of suspension O, and a is the distance between this point and its centre of gravity.

The above equation $I=maL$ gives a simple means of experimentally determining the moment of inertia of any body about any axis. Hang it up by this axis and measure a, the distance from it to the centre of gravity; then set it swinging slightly, and observe the length of a simple pendulum which keeps time with it: multiply the product of these two lengths by the mass of the body (in lbs. or grammes), and you have its moment of inertia under those circumstances.

81. *Centre of Oscillation or Percussion.*—A point O′ in a swinging body, situated in OG produced, at a distance L (the length of the equivalent simple pendulum under the circumstances) from the centre of suspension O, is called the *centre of oscillation*, because the body oscillates as if a portion of its mass were concentrated there, the rest of it being at O, in such a way that G remains the centre of gravity; this may be verified as regards mere time of swing by the experiment of swinging the body and a simple pendulum or plumb-bob together, and observing that, when of the proper length, the motions of the simple and of the compound pendulums are identical. It may be shown that the body will swing in just the same period if suspended at this point O′ as if it were suspended at O. This depends upon the fact that if I be its moment of inertia about a point O, at a distance a from G, and I′ its moment of inertia about a point O′, at a distance a', such that $a+a'=L$, then $I:a=I':a'$; or the length L is the same for both points.

A line through O′, perpendicular to OG, and to the axis of suspension, is sometimes called the 'line of percussion' or the *centre of percussion*, because this is the place where the body strikes things best without any jar on its support. A cricket-bat drives the ball best if the ball strikes it at a point on this line, and it does not then jar the hand.

To find all the points about which a rigid body (say for instance a flat plate or board pierced by a pin which supports it, fig. 23a) will swing in the same time as about any point O, determine experimentally the length of the equivalent simple pendulum, say by adjusting a plumb-bob to swing in the same time as the plate, and measuring its length; then mark this length upon the plate, as OO′, and draw a couple of circles with centre G, through O and through O′ respectively; the plate will swing in the same time if pivoted *at any point on either of these circles.* If pivoted at an axis between the circles, it will swing more quickly; if pivoted inside the smaller circle or outside the bigger circle, it will swing more slowly. The experiment should be tried; and if some care is bestowed upon it, and a series of pairs of such circles recorded, the result will be instructive to a student. The product of the radii of every such pair of circles will turn out the same; and, when multiplied by the mass of the body, it will represent the principal moment of inertia of the body—namely, that about the centre G.

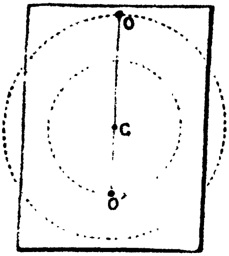


Fig. 23a.

The circumstances of a swinging body pivoted on a line or axis at any point A may be stated in terms of figure 24, where G is the centre of gravity and AKB is a right angle (or semicircle).

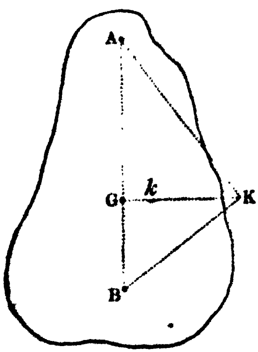


Fig. 24.

The moment of inertia of the body, about an axis parallel to the pivot but drawn through G, is the mass multiplied by GK^2. The moment of inertia about the pivot A is the mass multiplied by AK^2, and that about B is the mass multiplied by BK^2.

A is the centre of suspension, B is the corresponding centre of oscillation, and a line through B perpendicular to AB is the line in which a blow must act to spin the body automatically about A without any force from the pivot.

The length of the equivalent simple pendulum is AB, and the axes A and B are interchangeable.

Expressing these facts algebraically:

$$L = a + b, \qquad k^2 = ab, \qquad R^2 = a^2 + k^2;$$

R is called the swing-radius of the body about the axis A, and is such that $I_A = mR^2$ while $I_G = mk^2$.

The whole dynamic and static behaviour of the body is like that of two heavy particles rigidly connected, the one at A and the other at B; the mass of the one at A being $\frac{b}{a+b}m$, and the mass of the one at B being $\frac{a}{a+b}m$, so that their centre of gravity is the same as that of the body.

82. **Ballistic Pendulum.**—A heavy block of wood hung up as a pendulum by two strings, so that it can swing without any rotation, is sometimes used to measure the impulse (mv) of a blow, such as that of a rifle-bullet fired into the wood. The block will be displaced and will rise a vertical height, h, which must be observed (either directly or by calculation from the angle of swing); and the velocity v' imparted to it is calculated as $\sqrt{(2gh)}$. The velocity v with which the rifle-bullet struck the wood can then be found, if the mass m' of the block is known, from the equations,

$$mv = (m + m')v'$$

and

$$v' = \sqrt{(2gh)}.$$

EXAMPLES—XIV.

(1) A uniform rod 3 feet long is swung as a pendulum about one end. Find the length of the equivalent simple pendulum.

(2) Find the point about which the above rod should swing so that the time of oscillation may be a minimum, and find that minimum time.

In this case the centres of suspension and oscillation must be equidistant from the centre of the rod—see end of sect. 81.

(3) A uniformly thick rigid door on smooth hinges is shot at with a bullet. Find where and how the bullet must strike the door so as to cause no jar on the hinge.

(4) A one-ounce bullet fired horizontally into a 20-lb. block of wood suspended by two strings displaces it so as to rise 3 inches. Find the speed of the bullet.

(5) A 1-foot sphere hanging by a 6-inch string oscillates like a pendulum. Find its time of swing and the length of the equivalent simple pendulum.

(6) Find the period of a rectangular thin plate 4 feet × 3 feet swinging in its own plane about one corner.

(7) Find where else it may be suspended to swing equally fast, and find how it must be hung to swing fastest.

(8) A triangular plate of height h and mass m swings about its base. Find the length and mass of the equivalent simple pendulum.

(9) The same plate swings about an axis parallel to its base, bisecting the sides. Find the length and mass of the equivalent simple pendulum.

(10) Show that a triangular plate has the same moment of inertia about any axis as a system of 3 equal particles (each = $\frac{1}{3}$ of the mass of the plate), situate at the middle points of its sides, and find the time of swing of a triangular plate, with sides each 4 feet long, swinging in its own plane about a vertex.

(11) Find the mass of the equivalent simple pendulum in Ex. (1), if the mass of the rod is 4 lb.

(12) A chair weighing 20 lb. is hung by a point $2\frac{1}{2}$ feet from its centre of gravity, and is found to oscillate in precisely the same way as a simple pendulum 3 feet long. Find the moment of inertia of the chair about the point of suspension.

(13) Find the time the chair would take to complete a small oscillation.

(14) A one-ounce rifle bullet is fired into a suspended block of wood weighing 30 lb. If the blow causes the wood to rise a vertical height of $1\frac{1}{4}$ inches without any rotation, find the velocity of the bullet just before it struck the wood.

CHAPTER VI.

WORK AND ENERGY.

83. The present chapter is to indicate a method of treating the effects of force on matter in a perfectly general manner; all consideration of how the force acts, or what it acts on, being regarded as accidental and secondary. Whether the body acted on is a particle, or a rigid solid, or an elastic solid, or a liquid, or a gas, matters nothing; and whether the effect produced is motion, or strain, or both, or neither, also matters nothing. It is to treat of the effects of force in general on any body whatever.

84. Now, in order that an agent exerting a force may produce any effect on the body to which it is applied, it is necessary that the body shall yield somewhat—that is, that the point of application of the force shall move in the direction of the force; and whenever this happens—whenever the point of application of the force does move along its line of action—*some* effect is *necessarily* produced. Thus either the body is set rolling, or swinging, or moving in some way, or its motion is checked, or it is squeezed into smaller compass, or bent out of shape, or it is lifted up against gravity, or it is merely shifted along against friction, or it is warmed or electrified; no matter what the effect is, some effect is always produced, and the force, or more properly the agent exerting the force, is said to have *done work*. Moreover, a body upon which work has been done is found to have an increased power of doing work itself—that is, of producing physical changes in other bodies; and it is therefore said to possess more *energy* than before. This

increase of energy is indeed the most essential part of the effect produced in a body by an act of work.

85. **Energy** therefore is that part of the effect produced when work is done upon matter which confers upon the body possessing it an increased power of doing work.

86. The work done in any case is proportional both to the magnitude of the force and to the distance through which its point of application moves in the direction of the force. Unless the point of application moves, no work is done and no energy is produced, however great the force may be; for instance, a pillar supporting a portico is doing no work, though it is manifestly exerting great force.

Work, then, is the act of producing an effect in bodies by means of a force whose point of application moves through a distance in its own line of action, and it is measured by the product of the force into the distance, or*

$$W = Fs.$$

Such a force is conveniently called an 'effort' if the motion is in its own direction, and a 'resistance' if the motion is against it. (If the force and motion are inclined to each other, only one component of the force is the effective effort.)

The work is reckoned positive, and is called simply 'work,' when the body acted on is moved in the same sense as the force; if, however, by any means, it be caused to move in *opposition* to the force exerted by an agent, the work done by that agent must be reckoned negative—that is, work is done upon it.

Thus if a force of five poundals acts through a distance of six feet in its own direction, it does thirty times the work which would be done by one poundal acting through one foot. This latter work may be called a foot-poundal, and represents the F.P.S. absolute unit of work.

British engineers use as their practical unit the work

* The *moment of a force* was also defined as a force multiplied by a distance, but by a distance measured *at right angles to the force*. It is therefore an entirely different thing from work.

done by an effort equal to the weight of a pound acting vertically through a foot, and they call this a foot-pound. It is of course equal to 32 foot-poundals approximately. French engineers use the kilogrammetre as their practical unit of work, being equivalent to the weight of a kilogramme raised a metre. It is equal to 98,100,000 absolute C.G.S. units.

The C.G.S. unit of work is called an *erg* (from ἔργον, work), and is the work done by a dyne effort acting through a centimetre. There are 421,390 ergs in one foot-poundal. Since the erg is so small a unit, it has recently become customary to employ ten million ergs as a more convenient unit of energy for some purposes, and to call it a *Joule*. It is approximately $\frac{3}{4}$ of a foot-lb. The multiplicity of units is admittedly troublesome to present-day students, but it is one of the penalties they pay for living in an age of transition and activity.

The algebraic expression or 'dimensions' of an absolute unit of work is $\frac{\text{pound (foot)}^2}{\text{(second)}^2}$ or $\frac{\text{gramme (cm.)}^2}{\text{(sec.)}^2}$; being essentially a momentum multiplied by a velocity, or, what is the same thing, a mass multiplied by an acceleration and a length. The gravitational unit, or foot-pound, is a *weight* multiplied by a vertical height, and is frequently convenient; though it is liable to be hastily misinterpreted as an incomplete specification, the 'dimensions' of acceleration necessarily involved in it $\left(\frac{\text{foot}}{\text{sec.}^2}\right)$ being ignored, instead of being only taken as *suppressed* or 'understood.'

87. The effects produced in material bodies when work is done upon them are various, and constitute the different *forms* of energy. The full discussion of the subject of energy belongs to the science of physics, so we can here only just roughly enumerate its principal forms.

(1) Motion (whether translation or rotation).

(2) Strain (whether extension, compression, or distortion).

(3) Vibration (including the particular kinds called Sound). (4) Heat (sensible and latent).

(5) Radiation (including the particular kinds which are able to affect the eye, and which are therefore called Light).

(6) Electrification. (7) Electricity in motion. (8) Magnetisation.

(9) Chemical separation. (10) Gravitative separation.

To these we ought perhaps to add vital energy, only that it may be held to be included under head 9. It is quite possible that many of these may reduce to simpler forms; in fact, all but Nos. 9 and 10 are already pretty well known to be special cases of Nos. 1 and 2 (cf. sect. 5).

It is usual to consider those forms of energy which are more directly connected with large and visible masses of matter as more particularly the province of mechanics; and we shall here discuss only these more mechanical forms of energy, Nos. 1, 2, and 10.

The essential nature of No. 10 is at present unknown (see Introduction, sect. 3), but for most practical purposes it comes under the class indicated by No. 2.

88. Now the question arises—When work is done and energy produced, is it created out of nothing, or is it only manufactured from previously existing materials? The latter is the truth, for it has been found, as the result of innumerable experiments on the subject of 'perpetual motion' and others, that it is as impossible to create *energy* as it is to create *matter*, and that whenever energy appears as the result of work, it is always at the expense of some other form of energy which was previously existing. This fact is popularly expressed by saying that 'perpetual motion is impossible'—a statement which requires interpretation, because if there is one thing more universal than

another it is perpetual motion (see sect. 4). The statement, however, is understood to be an abbreviation for the following: It is impossible for us to construct any machine which shall move and do work (and therefore generate energy) of itself without consuming at least an equal quantity of pre-existing energy.

89. All this indeed, in a much more complete and accurate form—more complete, because it involves the *non-destruction* of energy as well as its non-creation—follows from Newton's third law of motion, sect. 52, provided we assume that energy is to be measured by work done—that is to say, that when a body does work, it loses a precisely corresponding quantity of energy, and that when a body has work done upon it, it gains an amount of energy equal to that work. For the third law tells us, that whenever force is exerted, and therefore (*a fortiori*) whenever work is done, two bodies are always concerned—there is the body which acts, and the body which is acted upon or re-acts—and these two bodies exert equal and opposite forces; hence whatever quantity of work one body does, the other has done upon it; or *the positive and negative works are equal* (see sect. 86, small print).

The 'agent,' or body which does the positive work, *loses* a certain quantity of energy. The body which has the work done upon it *gains the same amount*. Hence, on the whole —that is, taking both bodies into account—no energy is lost, and, algebraically speaking, no work is done. The energy is merely *transferred*, and the act of transfer involves two equal opposite works.

The law that, on the whole, no energy is ever created or destroyed by any forces which we know of and have experimented upon, is called *the law of the 'Conservation of Energy.'*

90. Just in the same way then that a force is the partial aspect of a stress, so work is the partial aspect of a some-

thing which consists of action and re-action, in the sense of work and anti-work, but which neither has, nor as yet perhaps needs, any name; and whenever we speak of 'work done,' it will be by *attending* to the action of one body on another, and neglecting the reaction of that other on the one. To summarise then: Work creates energy, anti-work destroys it, so both together simply transfer it. If it were possible to have a force without its anti-force, it would also be possible to get work done without its anti-work, but as a fact of experience it is *not* possible.

91. The fact that work is done whenever energy is transferred, taken in connection with the experience that energy often manifests a tendency to transfer *itself* from one body to another, and thereby to do work, has caused energy to be defined as *the power of doing work.* Now certainly a body possessing energy thereby possesses the power of doing an equivalent amount of work, *provided* the energy is of such a sort that it can be transferred to some other body; and in this sense *energy* and *power of doing work* are equivalent, though it is more precise to say that the possession of energy *confers upon a body* the power of doing work, than to say that energy *is* the power of doing work. It is quite possible, however, for a body to possess energy and yet have practically no power of doing work, for energy is not always *available.*

Thus, a stone lying on the ground may be said to possess an amount of energy corresponding to its fall to the centre of the earth, but this energy confers on it no power of doing work, for it would be impossible to let it fall without first expending a great deal more energy in digging a hole.

Again, energy is indestructible, and a given quantity may be transferred from one body to another, from A to B, from B to C, from C to D, and so on and back again, each time conferring upon its possessor a power of doing work, which work is done at each transfer by the body losing it. Hence, if it were correct to speak of work as being done by the *energy,* instead of by the body

possessing the energy, the working power of a given quantity of energy might be unlimited, and at any rate would be wholly incommensurate with the quantity of energy. The power of doing work, in fact, does not depend solely on the absolute quantity of energy in a body or system, but on its capability of being transferred to other bodies or systems. We seldom have to deal with the total or absolute energy in a body, but only with its variations.

92. There are, however, practical difficulties in effecting such a series of transfers of energy without loss of working power, for though the quantity is unalterable, yet the *quality* has a tendency to deteriorate.

These practical difficulties are very similar to those which you would experience if you attempted to transfer a given quantity of *water* down a series of vessels. For you might spill some, some would evaporate, some of the vessels might leak, and all would remain wet. The quantity of water would be unchanged—it would be all there—but some of it would be unavailable. It would be—not destroyed—only useless. Just so with energy, whenever it is transferred from one body to another—that is, whenever work is done, some of it is pretty sure to pass into a less available and more useless form. Its *quantity* is not altered, but its *availability* is less.

This tendency of energy to become less available is called the law of the *Dissipation or Degradation of Energy.* It may be expressed thus: When energy is *transferred* from one body to another, it is also always *transformed* from one of its forms to another, and some portion of the new form is pretty sure to be lower in the scale of energy than the original form; because of friction, imperfect elasticity, and so on. It is, in fact, impossible by any known process to raise energy in the scale of availability *on the whole.* Any given quantity, indeed, may be raised, but some other greater quantity will in the operation be degraded. The average is usually lower, and cannot be higher.

The energy of the earth in its orbit is not available to us. The energy of a flying molecule is almost unavailable, because we have as yet no means of dealing with molecules singly; if we could see and handle them, their motion would be as high a form of energy to us as the motion of other visible masses. Hence the distinction between high and low forms of energy is a purely relative one.

Energy falls in availability usually by becoming molecular—

that is, by being transferred from visible masses to their ultimate molecules. This transfer is effected by friction and viscosity; hence friction and viscosity are the chief practical causes of the dissipation of energy which is perpetually going on. No known means exist whereby energy is automatically *raised* in the scale of availability, but it has been surmised that perhaps this power appertains to certain forms of *life*.

93. Energy and work are not to be confounded together; and all such phrases as 'accumulated work,' 'conservation of work,' 'work consumed,' &c., should be eschewed, or else regarded as permissible colloquial inaccuracies. Energy is not work, but work can be got out of it if the proper condition be supplied. Energy might therefore be called *possible* work. For consider the two fundamental forms of energy:

(1) The free motion of masses of matter relatively to one another; and (2) The separation of masses of matter from one another against stress.

In the first case, the body possessing the energy is moving through a distance, but is not exerting any force. Supply a resistance, and work is immediately done. In the second case, the body possessing the energy is exerting force or pressure, but it is stationary. Allow it to move, and work is immediately done.

The two fundamental forms of energy, therefore, correspond to the two factors in the product called work—namely, F and s. The first form corresponds to s; there is motion through space, but no force. The second corresponds to F; there is force, but no motion.

The first is called **Kinetic Energy, or the energy of motion; the second might be called Dynamic Energy, or the energy of force (properly stress); or it might be called Static** Energy, to distinguish it from Kinetic. **As a matter of fact, however, it is generally called Potential Energy, which is not a bad name so long as it is not misunderstood**

to mean *possible* energy—a phrase without sense. Neither is Kinetic ever to be called **Actual** Energy. All energy is *actual* and *real*—potential just as much as kinetic; and both represent possible *work*—that is, work that will become actual as soon as the other factor is supplied. 'Possible work' merely means possible transfer of energy; just as money and goods, which might be called forms of mercantile energy, represent a possible transfer or exchange.

94. Whenever work is done, both factors, and therefore both kinetic and potential energy, must be present; and the energy is always passing from one of these forms into the other while the work is being done. For if the motion is *with* the force, the speed must increase, and if it is *against* the force, it must decrease; while in the first case the distance through which the force can act, or the *range* of the force, is decreasing, in the second increasing. The energy of a vibrating body is continually alternating from one form to the other.

Enough has now been said to show that the energy method of treating forces and their effects is a very general one, and extends to the whole of Physics. But the branch of the subject concerning which we can here enter into any detail will be a very small one, and will only extend to giving some examples of the transformation of energy from form 1, that of motion, to some other form, especially that of gravitative separation, and back again.

Measure of Kinetic Energy.

95. First consider how to measure the energy of motion in the case of simple translation of a particle; remembering that its energy (more strictly its *gain* of energy, over and above any other forms of energy, such as heat, &c., which it may retain constant all the time) is defined as equal to the work done by the force which caused the motion.

Now when a force F is applied to a mass m, the acceleration is

$$a = \frac{F}{m} \text{ (Chapter IV.)},$$

and the velocity generated when a body moves, from rest, a distance s, with the acceleration a, is given by

$$v^2 = 2as \text{ (Chapter II.), that is, } v^2 = 2\frac{F}{m}s;$$

an equation readily written in the form

$$Fs = \tfrac{1}{2}mv^2.$$

But Fs equals the work done by the agency of the force while it acts through the distance s; and as energy is measured by the work done in its production, it follows that the energy of a body of mass m moving with velocity v, is

$$\tfrac{1}{2}mv^2,$$

because v is the velocity generated in the body during the performance of the amount of work, Fs.

This expression, $\frac{1}{2}mv^2$, is a most important one, and it is called the **kinetic energy** of a particle due to its motion relatively to the body which is supposed to be at rest—usually, of course, the earth. It equals the work that has been done upon the body in setting it in motion, and also the amount of work which it must do in order to stop itself—that is, to transfer its energy to some other body, either to the earth or to anything else which happens to come in its way.

When one suspended elastic ball impinges directly on an equal one at rest, the first stops dead, and the other receives the whole motion; the energy has been here *obviously* transferred. The transference takes place just as really, though not so obviously, in every case where a body comes to rest or starts moving.

The unit of energy is the equivalent of the unit of work, and usually goes by the same name (sect. 86). For instance, the British unit of energy would be a foot-poundal, being the energy produced, or, rather, transferred, by the action of

unit force through unit distance; the C.G.S. unit of energy would be a dyne-centimetre, or one *erg*.

96. If a body, instead of being at rest when the force acted on it, had been moving with velocity v_0, it would have already possessed the energy $\frac{1}{2}mv_0^2$; and so the *gain* of kinetic energy, equivalent to the work done, would have been

$$Fs = \tfrac{1}{2}mv_1^2 - \tfrac{1}{2}mv_0^2;$$

where v_1 represents the final velocity possessed after the force has acted for a distance s. This immediately follows from the old equation $v_1^2 - v_0^2 = 2as$, if we write F/m for a, and leave the term Fs by itself on one side of the equation.

ILLUSTRATIONS.

97. A truck of mass 2000 lb. running along a level line at the rate of 20 feet a second, has an amount of energy equal to

$$\tfrac{1}{2} \times 2000 \text{ lb.} \times \left(\frac{20 \text{ ft.}}{\text{sec.}}\right)^2 = 400{,}000 \text{ foot-poundals},$$

or 12,500 foot-pounds. If it were required to stop it in a distance of 500 feet, we should have to apply a brake exerting 800 poundals, equivalent to a retarding load of 25 pounds-weight; for the work done by the truck against this force in the given distance would be 25 pounds-weight × 500 feet, or 12,500 foot-pounds, which is precisely the energy of the truck required to be destroyed, or rather to be transferred to something else.

One can always find the force necessary in any such case by dividing the work required by the distance given; for, of course,

$$F = \frac{Fs}{s}.$$

Again, to propel a one-ounce rifle-bullet ($\frac{1}{16}$th lb.) with a velocity of 1200 feet per second, will require work to be done upon it equal to the energy generated—namely,

$$\tfrac{1}{2} \times \frac{1 \text{ pound}}{16} \times \left(\frac{1200 \text{ ft.}}{\text{sec.}}\right)^2 = 45{,}000 \text{ foot-poundals},$$

or about 1400 foot-lb., or $\frac{5}{8}$ of a foot-ton. (This energy, and a good deal more, was contained in the charge of powder in the form

of chemical separation, No. 8 (sect. 87); a quantity is always wasted in the useless noise and flash attending the explosion, Nos. 3, 4, and 5.) This work must have been done by the powder while the bullet was travelling from the breech to the muzzle of the gun, a length of say four feet; hence the average force exerted by the powder must have been 45,000 foot-poundals divided by 4 feet, or 11,250 poundals, equivalent to about 3 cwt.

Suppose now in passing through the air it loses 400 of its velocity by friction, so that it reaches the target with the velocity of only 800 feet per second, then the energy of the blow will be

$$\tfrac{1}{2} \times \tfrac{1}{16} \times (800)^2, \text{ or } 20,000 \text{ units};$$

while that which has been 'lost' by friction (that is, transferred, some to the air and some to the molecules of the bullet, but in any case debased into the form of heat) is

$$45,000 - 20,000, \text{ or } 25,000 \text{ F.P.S. units of energy};$$

and this must be the number of units of work which have been done by the flying bullet against the resistance of the air. Hence if its *range*, or distance travelled, were 1500 feet, the average resistance exerted by the air must have been 25,000 divided by 1500, or $16\tfrac{2}{3}$ units of force, equivalent to the weight of about half a pound.

Finally, let a target stop the bullet dead in the space of $\tfrac{1}{4}$ inch ($\tfrac{1}{48}$th of a foot), then, since the whole (negative) work it has to do is numerically equal to the energy of the blow—namely, 20,000 units—it follows that the average force of the blow on the target is 20,000 divided by $\tfrac{1}{48}$, that is, 960,000 units of force, or about $13\tfrac{1}{2}$ tons weight, a much greater force than even the powder exerted; and this is apparent in the results, for the bullet is flattened out by the target, while the force of the powder had but a slight effect upon its shape.

Very likely an iron target would not yield so much as $\tfrac{1}{4}$ inch; if it only yielded half as much, the force of the blow would be doubled. Whether the bullet bounces off or not, matters nothing; it must have been stopped before its motion can be reversed. The reverse motion would not alter the *force* required to stop the bullet, but it would increase its *impulse* (sect. 48) by lengthening the time during which the force was exerted against the target. Thus, if the bullet bounced off with its original speed, the time and therefore the impulse would be double what they would have been if it had stopped dead like dough.

98. Notice the distinction between the *energy* of a blow, the *impulse* of a blow, and the *force* of a blow.

The energy equals Fs, or $\frac{1}{2}mv^2$.

The impulse equals Ft, or mv.

The average force equals F, or $\frac{\frac{1}{2}mv^2}{s}$, or $\frac{mv}{t}$ (cf. sect. 45).

It will be a good exercise to find from this last equation, in all the above cases, the time taken to do the work—that is, to transfer the energy. For instance, find the time of flight of the bullet, and also how long it took to travel the length of the gun, and so on. It is worth noticing that work

$$
\begin{aligned}
&= \text{force} \times \text{distance moved} \\
&= \text{impulse} \times \text{average velocity} \\
&= \text{momentum generated} \times \text{average velocity} \\
&= (mv_1 - mv_0) \times \tfrac{1}{2}(v_1 + v_0) \\
&= \tfrac{1}{2}m(v_1^2 - v_0^2).
\end{aligned}
$$

These two expressions for an average force, $\frac{mv}{t}$ and $\frac{\frac{1}{2}mv^2}{s}$, are worth comparing. The first we know is expressed in words by saying that force is rate of change of momentum; *rate* here having a reference to time, and meaning the increase per second of time elapsed. Similarly the second may be expressed by saying that force is rate of change of energy, only *rate* here has a reference to distance, and means the increase per linear foot of distance travelled.

The whole subject of the rates of variation of things with respect to different variables, considered as a branch of pure mathematics, is called the differential calculus, a science the foundation of which was laid by Newton and developed by Leibnitz for the purpose of treating questions concerning velocity, acceleration, and the like.

Measure of Potential or Dynamic Energy.

99. Now let us consider how to measure potential energy, or the energy of stress, especially in the form of gravitative

stress exerted between the earth and a raised weight. This is a very simple matter, for, supposing a stone is at a height h, we have a constant force mg exerted on the body, and a distance h through which it can act, so the work it can do while the stone falls is simply mgh; wherefore mgh measures the energy due to the relative position of the earth and stone, and the numerical value of this expression is often called 'the potential energy of the raised weight.' It equals the number of units of work that have been done upon the weight in raising it, and also the amount of work it must do whenever it drops. The energy is often *called* that of the weight, but it really belongs to whatever agent is exerting the stress pressing the weight and earth together (see Introduction); and as the nature of this agent is unknown, it is better not to speak of the potential energy *of* anything.

100. The gravitative energy of a pound of matter one foot high is called a foot-pound, because it is the effect which has been produced by a force of one pound-weight acting through a foot; it equals thirty-two F.P.S. units of energy or foot-poundals, because the weight of a pound equals thirty-two F.P.S. units of force or poundals. The unit of work or energy about corresponds to the raising a half-ounce weight one foot high (cf. sect. 46), (half an ounce being the $\frac{1}{32}$d of a pound).

'Thirty-two,' of course, stands for the value of g, whatever it may happen to be: it is different in different latitudes, and not necessarily exactly thirty-two anywhere. In French measure the numerical value of g is 981 (sect. 65); so the gravitative energy of a gramme of matter one centimetre high (called a gramme-centimetre) is 981 ergs, because the weight of a gramme is 981 dynes.

101. To keep a raised weight still, it must be supported, and it will exert pressure on its support, because it is being pressed by something towards the earth. This something

is not, however, yet doing work. Remove the support, and immediately the weight begins to move; hence now work is done, and the potential energy of the agent which exerts the pressure is transformed gradually into kinetic energy, and transferred gradually to the moving mass—to the weight itself, if falling freely—to whatever strings and wheelwork it is connected with, if it is constrained to fall slowly like a clock weight. When half-way down, the energy is half kinetic and half potential; when $\frac{3}{4}$ down, it is $\frac{3}{4}$ kinetic and $\frac{1}{4}$ potential, and so on.

For the original energy was mgh; but when half-way down, the potential energy is $mg\frac{1}{2}h$, or only half what it was, so the kinetic must be equal to the other half. When $\frac{3}{4}$ down, the potential energy is only $mg\frac{1}{4}h$, and the remainder is kinetic.

When within an ace of the ground there is no potential energy, and therefore the body has kinetic energy $\frac{1}{2}mv^2$, equal to the original energy, mgh.

This equation,

$$\tfrac{1}{2}mv^2 = mgh,$$

gives us the velocity acquired by a body freely falling a height h, as $v = \sqrt{(2gh)}$; a fact we knew perfectly well before, only we formerly arrived at it in a different way (see sects. 21 and 65).

The instant the falling body touches the ground compression occurs, and so work is done again, though this time very rapidly; and the energy is again transformed, and transferred, some to the molecules of the earth and ball as heat, some to the air in the form we call sound; while the rest, after having existed for an instant as strain or stress energy between the earth and ball, reappears as kinetic energy in the bouncing ball. No ball, however, is perfectly elastic, so after a few bounces it will come to rest, and will possess neither kinetic nor potential energy relatively to the earth (it will be a little hotter than it was—that is all). To raise it again, something else must do work upon it.

102. As another illustration, consider a body sliding down

a rough inclined plane. Let a mass m slide from A to B (fig. 25), a length l, against a force of friction f, the *vertical* descent being h. Then the work done against friction is fl; the work done upon the mass is $\frac{1}{2}mv^2$ if it reaches B with the velocity v; and all this work has been done by gravity. But the work done by gravity is the force mg multiplied by the distance moved through *in its own* (*vertical*) *direction*—namely, h; so we have the equation,

$$fl + \frac{1}{2}mv^2 = mgh,$$

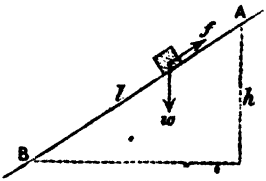


Fig. 25.

from which v can be readily found. The term fl represents the amount of energy which is transformed (degraded) into heat—that is, it is the 'mechanical equivalent' of the heat generated.

If $f=0$, that is, if the plane be smooth, the velocity v acquired in descending the vertical height h down the plane is the same as that found for a *freely* falling body in the last section, and it has no connection with the slope of the plane; showing that the *path* of a falling body has no influence on the *velocity* acquired by it, provided everything be smooth. (The *time* of descent is greatly influenced by the path.)

103. The simplicity of *gravitation* examples is due to the fact that the force acting (the weight of the raised body) is constant and does not alter as the weight descends. But in every case, if s be the *range*—that is, the distance through which the force can act—and if F be the *average* value of this force, the potential energy is Fs. (See Appendix, p. 307.)

Energy of Rotation.

104. So far we have only considered energy of motion in the form of translation, or the motion of a particle; but the energy of a rotating body can now be easily expressed, since

it is made up of particles, and the energy of the whole is the sum of their separate energies.*

Any particle of mass m, at a distance r from the axis of a body rotating with angular velocity ω, is revolving round and round a circle with velocity $v = r\omega$, and its energy is $\frac{1}{2}mv^2$, or, as it may be also written, $\frac{1}{2}mr^2\omega^2$. Now the energy of the whole body is the sum of the energies of all the particles in it; it is therefore

$$\Sigma(\tfrac{1}{2}mv^2) = \tfrac{1}{2}\Sigma(mr^2\omega^2) = \tfrac{1}{2}\omega^2\Sigma(mr^2);$$

for, since the ω is constant, it may be taken outside the sign of summation; but $\Sigma(mr^2)$, the sum of the second moments of inertia of all the particles in the body, is the quantity we have called *the moment of inertia* of the rotating body

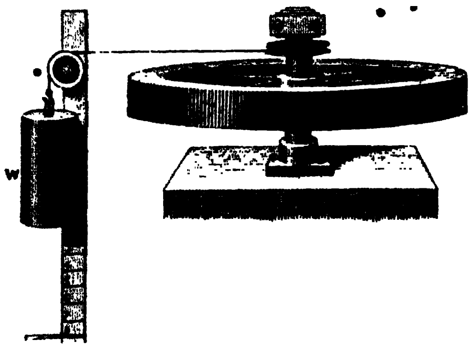


Fig. 26.

(sect. 43), and denoted by I; hence the simplest expression for the energy of a rotating body, like a flywheel, is

$$\tfrac{1}{2}\mathrm{I}\omega^2.$$

A flywheel mounted on a vertical axle can be started spinning by a descending weight, as shown in fig. 26.

* Notice that the parallelogram law (sect. 26) does not apply to the composition of *energies*. Energy is not a directed quantity, and simple arithmetical addition applies to it.

By the time a weight of 14 lb. over and above what is needed for friction has thus descended 6 feet, it will have done 84 foot-pounds of work and generated this amount of kinetic energy, of which a portion will belong to the wheel, and a portion to the weight. If it has taken 10 seconds to descend, its acceleration has been $\cdot 12$, and its kinetic energy is $14 \times \cdot 12 \times 6 = 10\cdot 08$ foot-second-units, or, what is the same thing, $\frac{1}{2} \times 14 \times (1\cdot 2)^2$. This is less than one-third of a foot-lb., and all the rest of the energy belongs to the wheel. If the winding pulley were $\cdot 6$ foot in circumference, it would now be making two revolutions a second, therefore its angular velocity would be 4π, and so its moment of inertia would be determined by the energy equation $\frac{1}{2}I(4\pi)^2 = 84 \times 32 - 10\cdot 08$.

105. A rolling sphere or cylinder has a motion compounded of a translation forward and a rotation round the centre of its rim, and consequently its energy is similarly compounded. Its translational energy is $\frac{1}{2}mv^2$, where v is the speed of its centre forward; and its rotational energy is $\frac{1}{2}I\omega^2$, where I is the moment of inertia of the body about its centre; but a simple relation holds between v and ω, since the speed of rolling advance is the same as the speed the rim would have if the centre were stationary; wherefore the above two terms may be added together, making $\frac{1}{2}(I + mr^2)\omega^2$. This suggests that the body is really rotating at each instant about the point where its rim touches the ground, and that its moment of inertia about any point on its rim is greater than that about its centre by mr^2; at any rate these statements are true ones, as may be seen by referring to the list of Moments of Inertia, sect. 43.

EXAMPLES.—XV.

(1) **A body slides down a rough plane, travelling 20 feet along the plane, but only descending 12 feet vertically. If the force of friction is equal to ¼th of the weight**

of the body, find the velocity gained during the descent.

(2) What is the work that must be done in order to propel a 3-lb. stone at the rate of 40 feet a second?

(3) A simple pendulum is pulled aside till its heavy bob is raised 3 inches, and is then let go. Find its velocity when it passes its lowest point.

(4) What initial velocity is necessary to make a rifle bullet strike a target placed 300 feet high vertically above the gun with a velocity of 600 feet per second, neglecting the resistance of the air?

(5) What would be the answer to the last question if the bullet weighed an ounce, and if the resistance of the air were taken to be equivalent to a drag-back equal to the weight of 5 ounces?

(6) If a projectile were started in any direction with the velocity 80, and arrived at another point on the same level with the velocity 30, after having travelled 150 feet, what must the average resistance of the air have been equal to?

(7) What force would be necessary in order to stop the projectile of Question No. 2 in the space of 6 feet, and how long would it take?

(8) Find the mechanical equivalents of the heat generated by friction in the motions considered in Questions 1, 5, 6, and 7; assuming the mass to be 3 lb. in each case.

(9) What is the energy of a hollow globe 2 feet in diameter which is swinging round in a horizontal circle at the rate of 90 revolutions per minute, the mass of the ball being 5 lb., and the radius of the circle described by its centre 3 feet?

Consider the string so nearly horizontal that practically the ball rotates once during each revolution.

(10) What is the energy of a uniform steel disc a yard in diameter and an eighth of an inch thick rotating about a vertical axis 3000 times a minute, if a cubic inch of iron weighs 4 lb.?

(11) What is the kinetic energy of a 5-cwt. projectile moving with a velocity of 2000 feet per second?

(12) A body whose mass is 12 lb. moves from rest with a uniform acceleration of 100 inches-per-second per second. Calculate its kinetic energy after it has moved a distance of 20 feet.

(13) Find what work is being done per minute—that is, find the

activity or the power of an engine which is raising 2000 gallons of water an hour from a mine 300 feet deep.

A gallon of water weighs 10 lb.

(14) A train of 150 tons is running at 60 miles an hour. What force is required to stop it in a quarter of a mile?

(15) A small heavy body weighing 20 lb. slides down a rough circular arc 10 feet in radius whose plane is vertical. It begins to move from one end of a horizontal diameter, and is found to reach the lowest point with a velocity of 12 feet a second. How many foot-pounds of work have been done against friction during the motion? And if the same proportionate loss of energy occurred in the next portion of the same circle, how high would it ascend?

(16) A massive slow-moving flywheel, mounted on a horizontal axis 1 foot in diameter, possesses 1500 foot-pounds of kinetic energy, which is used to raise a weight of 25 lb. by means of a rope coiled round the axis. Assuming that a weight of 5 lb. is able to overcome the friction, how many times will the wheel revolve before it comes to rest? How many revolutions in the opposite direction must be made before the original energy is restored to the wheel?

(17) A railway carriage of 4 tons moving at the rate of 5 miles an hour strikes a pair of buffers which yield to the extent of 6 inches. Find the average force exerted upon them.

(18) A three-ton truck sliding down a plane rising 1 in 20 acquires a speed of 30 feet a second after travelling 500 feet down the plane. Find the average force of friction acting on it. What velocity would it have acquired if there had been no friction?

(19) A train of 50 tons moves up a rough incline of 1 in 10, the resistance caused by friction being 16 lb. weight per ton. What horse-power must the engine exert in order to maintain a uniform speed of 3 miles an hour?

A horse-power was defined by James Watt to mean 33,000 foot-pounds of work per minute.

(20) If a horse walking once round a circle 10 yards across raises a ton weight 18 inches, what force does he exert over and above that necessary to overcome friction?

(21) Calculate the work done in turning a wheel ten times round against a load of 15 lb. applied by means of a string

wrapped round its axle, which is 8 inches in diameter. If the wheel starting from rest makes 20 revolutions in the first quarter minute after being let go, what is its moment of inertia, supposing no friction?

(22) A mass of 224 lb. falls from a height of 10 feet upon a pile. Express its energy and its momentum when it reaches the pile.

(23) A man cycles up a hill, whose slope is 1 in 20, at the rate of 4 miles an hour. The weight of man and machine is $187\frac{1}{2}$ lb. What work per minute is he doing?

(24) At the top of the hill the cyclist is met by a strong head-wind, and he finds he has to work twice as hard to keep the same rate of 4 miles an hour on the level. What force is the wind exerting against him?

(25) If a train moving at 80 feet per second, up an incline of 1 in 64, slips a carriage, how far will the carriage move before beginning to run back, ignoring axle-friction?

(26) Find the horse-power of a locomotive which draws a train at 10 miles per hour up an incline of 1 in 40, the weight of train and engine being 400 tons.

(27) Find the horse-power of an engine which draws a train of 100 tons up an incline of 1 in 60 at a speed of 30 miles an hour, the friction being equal to a drag of 20 lb. weight per ton.

(28) A sledge left to itself slackens speed from 30 to 20 feet a second while going 15 yards. Assuming the coefficient of friction constant, find its value; also find how soon and in what distance the sledge will stop.

(29) A train weighing 60 tons, and running at the rate of 40 miles an hour, is stopped by an obstacle in 10 yards. What is the average force applied by the obstacle?

(30) How much work has a man, weighing 16 stone, done in walking twenty miles up a slope rising 1 in 40? What force could drag a dead load of the same weight up the same hill—(*a*) if the friction be negligible, (*b*) if the friction be $\frac{1}{4}$ of the weight?

(31) How much energy is expended in winding up the hour-striking part of a turret clock each day, if its weight of 2 cwt. descends 1 inch for each stroke of the bell, and if the friction is equivalent to $\frac{1}{4}$ of the weight? On the same hypothesis as to friction, how much of the energy is available for the production of sound?

(32) A sliding and a rolling body start down similar slopes together; which travels fastest, if one plane is smooth, and the other only just rough enough to ensure rolling?

(33) What must be the coefficient of friction of the plane previously smooth in order that the roller and the slider may travel at the same pace and arrive together? Take the case of a plane whose height is $\frac{1}{4}$th the base, and the rolling body a solid cylinder.

(34) Show that all solid and homogeneous spheres in rolling down an incline will travel together if they start together, whatever their size or material.

(35) Show the same for cylinders, and find by how much the spheres will beat the cylinders on a given slope.

(36) What is the energy of a pendulum bob weighing half a ton and swinging past its equilibrium position at the rate of 1 foot a second?

(37) What energy is stored in a cross-bow whose cord has been pulled 15 inches with a maximum force of 2 cwt.

CHAPTER VII.

COMPOSITION AND RESOLUTION OF FORCES.

(Introduction to Statics.)

106. Hitherto we have only considered the effect of a single force when it acts on a particle or on a rigid body, and we find that it may either pull the body along (translation), or turn it round (rotation), or do both at once. But in very few cases in practice do we have only one force acting in this way; often there are a great number of different forces, so that it becomes necessary to consider how the motive effect of a number of forces may be deduced. The simplest way is to reduce the forces in number; and the following statements may be recorded here as memoranda:

When any number of forces act on a *particle* they may always be reduced to one—that is, they may be replaced by a single force which produces precisely the same effect as them all. This single force is called the **resultant**; and the operation of reducing the number of forces is called the **composition** of forces.

If a number of forces act on different points of a *rigid body*—that is, an assemblage of particles connected rigidly together—they cannot in general be reduced to one force, but they may always be reduced to two forces in different planes (sect. 121). They can, however, always be reduced to a single force (of which what is called 'a couple' is a special case) if they either all pass through the same point (that is, virtually act on a particle), or else all lie in the same plane (cf. sect. 137).

A picture hanging by a cord over a nail furnishes us with an example of a rigid body acted on by several forces, and the tension in the two parts of the cord is equivalent to the weight of the picture. A weight resting on a tripod-stand is another example, and the three stresses in the legs are equivalent to the one weight. Again, a table or chair is supported by as many forces as it has legs, unless some are too short (which they often are). A teetotum is spun by forces which may be reduced to two equal and parallel ones in opposite directions, and we have here a case of pure rotation without translation. A kite in the air is acted upon by the wind pressing it, by a tension in the string, and by the pull of gravity; and the kite moves about according to the direction of the resultant of all these forces.

107. Again, for some purposes, it is convenient to analyse or split up a single force acting on a body into two or three components so as to study their effects separately. This operation is called the **resolution** of forces; and it is carried out in the same way, and for the same sort of object, as the resolution of motions and velocities (see sect. 30). Thus, suppose a body resting on an inclined plane, we may resolve its weight into two forces, one perpendicular to the plane, and therefore balanced by its resistance; the other acting along the plane and producing motion, except in so far as it is balanced by friction. Again, in a windmill, it is convenient to resolve the wind's pressure on the sails into two components—one the effective one in the direction of motion; the other a useless one in the direction in which, by the construction of the machine, no motion is allowed.* This last component, therefore, only produces strain.

Composition of Forces acting on a Particle.

108. The method of compounding forces into a resultant, or resolving them into components, is a very simple one, being the same as that by which motions were compounded and resolved. For if several forces act on a particle, each tends to accelerate its motion in its own direction, and the resultant acceleration is the resultant of these several com-

* *N.B.*—A windmill always faces the wind.

ponent accelerations. The single force which would cause this resultant acceleration is equivalent to the several forces combined, and is called the resultant force; while the separate forces are called its components.

Hence forces are compounded in the same way as accelerations or motions (sect. 28), being represented by lines in their respective directions proportional to the accelerations they could produce in unit mass. And they can be resolved also in the same way. No further proof of the triangle or the polygon of forces is necessary.

The above deduction from Newton's second law, $F = ma$, does not, however, establish the *position* of the resultant, but in the case of a particle this is obvious. The further condition necessary for an extended body is given in sect. 119.

The rule, then, is—Draw a set of lines one after the other, without taking the pen* off, parallel to, and in the same sense as the successive forces acting on the body, and proportional to them in magnitude; then the line required to complete the polygon, taken in the reverse sense (that is, drawn *from* the starting-point, not *to* it), will be the resultant in magnitude and direction. The forces may be taken in any order just as the motions might (sect. 24).

Since we are only dealing with a particle, this is the full and complete solution; for the resultant, of course, acts on the particle, and therefore its position is known; and the three things, magnitude, direction, and position, completely specify a force (see sect. 51).

The resultant of *two* forces is often more conveniently expressed as the diagonal of the parallelogram whose sides represent the forces, than as equal to the third side of a triangle.

109. **Examples of the Composition of Two Forces.**—A particle of mass m is pulled along by two strings—one

* If the forces do not all lie in one plane, the polygon cannot be drawn on paper, but it may be constructed in wood.

always pulling east, with a force P; the other always north, with a force Q. What is the acceleration and direction of motion?

Drawing the two forces P and Q (fig 27), one finds the resultant R at once as equal to $\sqrt{(P^2+Q^2)}$ by Euclid, I. 47; and since this

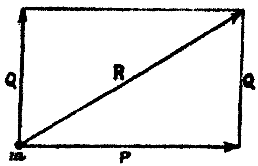


Fig. 27.

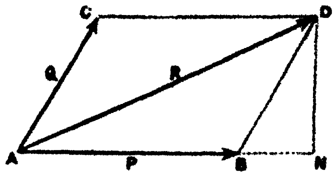


Fig. 28.

is the resultant force, the acceleration is $\frac{R}{m}$ along the diagonal of the parallelogram. If the two forces P and Q were equal, then R^2 would be simply $2P^2$; that is, $R = P\sqrt{2}$, a result worth remembering.

Suppose now that the two forces act at some acute angle, say 60°, then to find R we may use Euclid, II. 12, which says that AD^2 exceeds $AB^2 + BD^2$ by twice the rectangle $AB \cdot BN$ (fig. 28).

The angles CAB and DBN are always equal (I. 29), and if each equals 60°, BN is easily seen to be half BD, because the triangle BND is then half an equilateral triangle; so putting in this value for BN, and noting that $BD = AC = Q$, and therefore $BN = \frac{1}{2}Q$, we can write the general relation $AD^2 = AB^2 + BD^2 + 2AB \cdot BN$ in the form $R^2 = P^2 + Q^2 + PQ$ for the case when the angle between P and Q is 60°.

If the angle CAB between the forces had been an obtuse angle, such as 120°, we should have proceeded similarly, only using Euc. II. 13, and we should have arrived at $R^2 = P^2 + Q^2 - PQ$.

Similarly we might proceed for angles between P and Q of 45° or 135°, of 30° or 150°; but for angles in general, though the relation

$$AD^2 = AB^2 + BD^2 + 2AB \cdot BN,$$

will always apply—regard being paid to sign in the last term (see sect. 12)—yet it is not so easy to express the side

BN in terms of the side BD (or Q)—the subject of the mensuration of triangles, or Trigonometry, not being supposed known at this stage.

Our resource is then to find the resultant by construction; and this indeed is often a very good way, even when one knows some trigonometry. You lay off on paper the two given forces to any scale, and inclined at the proper angle; then you complete the parallelogram, and measure the diagonal on the same scale—this gives you its magnitude; and its direction referred to the given forces you get also from the figure.

110. Notice that in the parallelogram of forces, you really have two diagrams drawn together as one: a representation of the forces, and a geometrical construction; but they should be understood to be essentially distinct. The proposition of the triangle of forces is really the geometrical part of the parallelogram by itself.

An example will render the meaning of this clearer. Let two forces, 6 and 8, act on a particle with an angle of 60° between them. Find their resultant.

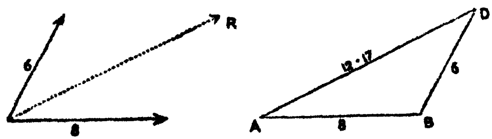


Fig. 29.

On the left of fig. 29 is a picture of the given forces. On the right is the geometrical figure—namely, a triangle in which AB represents the force 8, BD the force 6, and AD their resultant, in magnitude and direction. (AD equals 12·17 nearly, as may be found either by drawing and measuring, or by calculating it from sect. 99 as $\sqrt{(8^2+6^2+8\times 6.)}$ Its position is known, for of course it acts on the given particle; so we return to the left-hand diagram, draw through the point of intersection of the two given forces a line equal and parallel to AD, and this will be the resultant. Obviously it is the diagonal of the parallelogram of forces—the triangle ABD is simply half the parallelogram; compare fig. 28.

Observe that the geometrical construction is based upon only magnitude and direction: it does not give you position; this must always be determined from the positions of the given forces in the force diagram. It is not usual to separate the two figures in simple cases, but as a matter of principle it is best always to keep them distinct.

111. The diagrams for one instance of the polygon of forces may be also given, just to make sure it is fully understood.

Forces in a plane, of magnitudes 4, 5, 3, 8, act on a particle, their directions making angles with each other of 75°, 45°, and 120° respectively. Find the resultant. In actual cases the

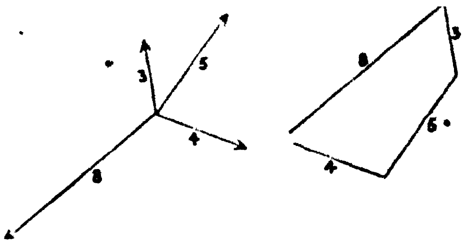


Fig. 30.

angles between the forces are not specified numerically, but are indicated directly. In artificial questions, however, like the above, when the angles are specified in degrees, a protractor may be used to lay off the directions.*

It turns out to be ·504, so that the given forces are very nearly in equilibrium, or their resultant is very small.

Observe the reciprocity of these diagrams. In one the lines meet in a point, in the other they enclose an area.

Try drawing the sides of a polygon in some other order, and

* In all these figures the lines are drawn *parallel* to the forces; this is the easiest, though not the essential plan. What *is* essential is, that the lines shall represent the direction of the forces in some understood manner. It is usually said that they will do either parallel or perpendicular; but they would do equally well if all were inclined at 45°, or at any other angle, to the forces which they respectively represent, provided this angle were the same for all. The interior angles of the polygon are supplementary to the angles between the corresponding forces.

see that you always get the same result. Especially try the order 4, 3, 8, 5; for the polygon then happens to be a crossed one.

The bits of forces represented by the lines surrounding the enclosed space in a crossed polygon are in equilibrium (Chapter VIII.), and may be removed from the particle without disturbance.

In the above case the enclosed bit will be found to be an equilateral triangle, and the forces which may be removed are three threes—namely, three parts from force 8, three from 4, and all of 3; the forces left being 1, 5, 0, 5. Construct the polygon for this mutilated set, and see that you still get the same resultant. Notice the reason why the three removed forces were in equilibrium—namely, that they were equal and lay symmetrically, making angles of 120° with each other.

You are strongly recommended at once to get out your instruments and a sheet of drawing-paper, and verify all this by careful drawing, as well as some of the examples in Ex. XVI. The instruments needed are a graduated scale of equal parts, a couple of set squares to act as a parallel ruler, perhaps also a T square, and a pair of compasses. A protractor for measuring angles is also convenient.

Resolution of Forces.

112. Every force may be split up into two definite components acting at given angles with it; but, if the angles are *not* given, a force may be resolved into two components in an infinite number of ways; in other words, the same line may be the diagonal of an infinite number of parallelograms (fig. 31). One chooses in each problem the particular pair of components which are most convenient, the most convenient being usually at right angles to each other. Often one is in the direction of possible motion,

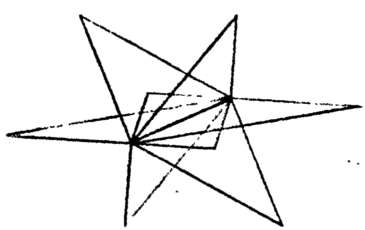


Fig. 31.

and the other perpendicular to it; again, in cases where gravity is concerned, one is often horizontal and the other vertical.

Verify, by drawing, the following: A force of 8 units is equivalent to two components of $\frac{8}{\sqrt{(2+\sqrt{3})}}$ each, acting one on each side the given force at angles of 15° with it; also to two of 8 each, if the angles be 60°; also to two of $\frac{8}{\sqrt{(2-\sqrt{3})}}$ each, if the angles be 75°; also to a component 4, acting at an angle of 60°, and another of $4\sqrt{3}$ at an angle of 30°; and so on.

$$(\sqrt{2}=1\cdot4142\ldots\ldots;\ \sqrt{3}=1\cdot732\ldots\ldots)$$

113. **Constrained Motion.**—When a body is constrained, as by a line of rails, to move in some fixed direction, and when the propelling force acts partly athwart the constraint, its treatment is simplified by resolving it into two components, such that one acts along the line of possible motion, and the other across it. The first is the effective or working force, and it either accelerates or retards the motion, while the other is the lateral force exerted against the rails and balanced by their constraining pressure.

114. To illustrate the use of this, take a mass m, or say $\frac{1}{4}$ lb. for those who like numbers best, on a smooth inclined plane inclined to the horizon at an angle say of 30°. Such a body is constrained by the plane not to fall vertically, and the forces acting on it are the force w $(=mg)$ due to gravity acting downwards, and the pressure of the plane, say R, acting normal to the plane. Now, resolve the downward force into two—one along the plane, that is, in the direction of motion, and call this effective component p; the other normal to the plane, as q (fig. 32).

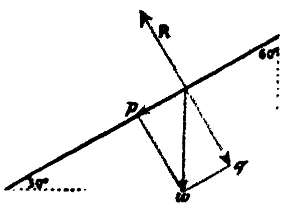


Fig. 32.

This you do by drawing a parallelogram with sides in these directions, and of such size that w (in the present example, $\frac{1}{4}g$ or 8) is the diagonal. The angle between p and w is 60°, that between w and q is 30°, and we have just found (end of sect. 112) that a force of eight units is equivalent to two forces, 4 and $4\sqrt{3}$, acting at 60° and 30° respectively; so then $p=4$ and $q=4\sqrt{3}$.

The motion is in the direction of p, the acceleration being $\frac{p}{m}$ or $\frac{4}{\frac{1}{4}} = 16$, or half its unconstrained value; whereas there is no motion in the direction of q, because it is balanced by the constraining force R, hence $R = q = 4\sqrt{3}$.

115. **Friction.**—The reaction of rough surfaces will afford us other examples. When anything exerts pressure on a plane surface, the reaction of the surface is in general inclined in some direction or other to the surface—usually in that direction most likely to oppose relative motion; but it is convenient to resolve this reaction into two—one normal to the surface (normal merely means perpendicular), which is called the **normal pressure**; the other along the surface, which is called the **friction**. If either surface be perfectly *smooth*, this last component is absent, and all the reaction is normal. And even for rough surfaces this may be so too, as in the case of a ball resting on a level floor; but if any forces are tending to cause motion over rough surfaces, then there is some component along the surfaces, or friction, which always opposes the motion.

The force of friction is precisely equal and opposite to the resultant force tending to cause the motion, so long as the body does not move; but if the applied force gradually increases, it will, at a certain instant, become too much for the friction, which reaches a maximum, and can increase no further; so then motion ensues, the effective or accelerative force being the applied effort, minus the friction. For instance, in the above example of the inclined plane, suppose the force of friction to be called f, it would act up

the plane in exact opposition to p. If the body were at rest, it would be so because $f=p$; if it were in motion, the acceleration would be $\frac{p-f}{m}$.

It is found experimentally that the maximum or critical value of f is proportional to the normal pressure R between the surfaces, the ratio between f and R depending on the nature of the surfaces in contact, and being called the **Coefficient of Friction.***

Say that this coefficient in the above case of the inclined plane is $\frac{1}{8}$, so that $f=\frac{1}{8}R$, then the acceleration would be

$$\frac{4-\frac{1}{8}4\sqrt{3}}{\frac{1}{4}}=16-2\sqrt{3}.$$

The actual pressure or reaction between the surfaces in contact is, of course, the resultant of the two forces R and f, the normal pressure and the friction—that is, it is the square root of the sum of their squares; and the angle which its direction makes with the normal when the two surfaces are on the point of sliding over one another is called the *limiting angle of friction.*

When a body is *resting* on an inclined plane, supported only by friction, the resultant reaction of the plane must be *vertical*, in order to balance the weight; the greatest tilt that can be given to the plane without causing slip is therefore equal to the limiting angle of friction—that is, the maximum angle between vertical and normal; and so this angle is called the angle of repose. On any plane with less tilt than this there will be a margin, since the reaction is not as

* The coefficient of friction when the surfaces are in actual relative motion is usually less than when they are just going to move; hence there are two coefficients of different value, one when the bodies are on the point of slipping, called the static friction, or 'stiction;', the other when the surfaces are actually sliding over each other, called the kinetic friction. The latter is the smaller of the two, and frequently depends somewhat upon the speed of the relative motion. Lubrication not only lessens friction but tends to abolish the difference between static and kinetic friction. Resin, on the other hand, exaggerates this difference. Vibrations, with accompanying noise, are liable to be set up whenever there is a marked difference between static and kinetic friction, because the slipping is apt to become intermittent, being alternated with moments of adhesion.

much inclined to the normal as it can possibly be (see, for further information, sect. 142).

116. It is often convenient to resolve motions and velocities. Thus, as we saw (sect. 76), a projectile shot up at any angle has a certain initial velocity imparted to it, which may be conveniently resolved into two—one a horizontal one unaffected by gravity, which therefore remains constant except for the resistance of the air; the other a vertical one, which is gradually diminished by gravity at a definite rate, until it is converted into a negative, that is, a downward, velocity which increases at the same rate till the body strikes the ground.

Again, take a north-east wind. This may be considered as made up of a north and an east wind, each $\frac{1}{\sqrt{2}}$th of the actual strength, and on any thin, flat, smooth surface facing the north only the northerly component can exert any pressure, the easterly component simply gliding over it.

Or suppose the surface faced NNW., and we wanted to find the pressure on it; the wind might be resolved into an NNW. component, $\frac{1}{2}\sqrt{2-\sqrt{2}}$ times its strength, and an ENE. one $\frac{1}{2}\sqrt{2+\sqrt{2}}$ times its strength, and the surface would experience the pressure of the NNW. component only, the other being useless.

This is how one deals with kites and windmill- and boat-sails. They are all surfaces exposed in a skew fashion to the wind, so that the pressure on the surface is a component only of the whole available force of the wind. The sails of a windmill are set so as to be inclined both to the direction of the wind and to the direction of possible motion; so also usually are the sails of a boat. It is convenient to remark and remember that, disregarding viscosity or fluid friction, the pressure of a fluid is always normal to surfaces immersed in it.

117. In the case of a kite the normal pressure of the wind is balanced by two other forces, the pull of gravity and the pull of the string, otherwise the kite would be blown about, scarcely experiencing any pressure at all. The sails of a windmill are not blown in the direction of the normal pressure on them, but in some other direction determined by the way they are set on the axle and by the sole direction in which this can turn; the axle is always purposely set so as to face the wind, and so the sails can only move in a plane perpendicular to the wind. So also with a boat; the reason why it is not blown in the direction of the normal pressure on its sails is that it is more easily moved through the water lengthways than breadthways because of its shape. Hence the normal pressure of the wind requires again resolving into two components, one along the direction of easy motion, the other at right angles to it. The first component is the active one in the case of both windmill and boat; the other component is entirely counteracted in the case of the windmill, but in the case of the boat it does cause a slow broadside motion, which is called leeway.

Thus if BR (fig. 33) represents the plan of a boat, MS its sail, and W the relative direction and strength of the wind (represented also by the arrow i), P is the normal pressure and Q the useless

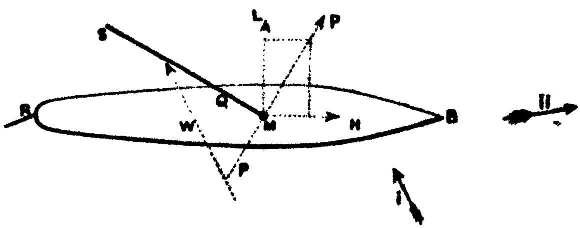


Fig. 33.

component or tail-wind. Producing P for convenience, and resolving it along and across the boat, H is the effective component producing headway, and L is the leeway component. The arrow

ii shows the direction in which the boat might tend to sail.* The rudder R is represented as turned in the direction required to counteract the leeway and make it sail along the line RB produced.

The *rudder* also affords an illustration of the present subject. When turned, there is a normal pressure on its front surface due to its motion through the water, and this pressure is resolvable into two forces—one in a direction opposite to the boat's motion, which simply acts as a drag (hence in racing, the coxswain uses the rudder as little as possible), the other at right angles to the length of the boat, which pushes the stern round.

It is obvious that no force can directly exert pressure at right angles to itself, and yet it is easy for a ship to sail at right angles to the wind. The reason is, that the sails act as a mediary, being inclined to both wind and boat. The force directly urging the boat is a component of the pressure on the sails, this pressure again being due to a component of the wind's motion. Remember that the effective wind—the wind felt by the ship—is the *relative* wind, which is compounded of the true wind and the speed of the ship (cf. sect. 31). This explains why a ship can sail very close to the wind.

These examples will serve to illustrate the application of the principle; but other examples occur daily, and may be worked out in the same way as the preceding cases, drawing and measuring being often sufficient.

EXAMPLES—XVI.

(1) Find the resultant of two equal forces each equal to 10 units for each of the following cases—namely, when the angle between them is 120°, 90°, 60°, 45°, 30°, respectively.

* This assumes that the sail is set amidships. In practice there is always a preponderance of sail towards the stern, consequently an unsteered ship gets blown round, and 'sails up into the wind's eye.' The rudder would therefore more likely have to be turned the other way, so as to counteract the action of the wind in causing rotation. The effect of the wind on the body of the boat has also to be taken into account in practice, and this may be different according as the bow or the stern is most out of the water—a thing which depends on the distribution of the load.

(2) Resolve the force 12 into two forces, making angles of 45° with the given force on either side of it.

(3) A picture weighing 12 lb. is hung by a cord over a nail so that each half of the cord makes 45° with the vertical. What is the tension in the cord?

(4) Find the tension in a picture cord when the two halves of the cord make an angle of 30°, 60°, 90°, 120°, 150° with each other. What is the way to hang a heavy picture so as to get the least tension in the cord?

(5) A weight of 10 lb. is placed on a smooth plane inclined 30° to the horizon. What force acting horizontally is required to support it? What force acting along the plane will suffice? and what is the normal pressure on the plane in each case?

(6) A carriage weighing 2 tons is to be drawn up a smooth road by a rope parallel to the road. The road rises 4 feet in a slope of 32 feet. What must the pull of the rope exceed in order that it may move the carriage?

(7) What weight can be drawn up a smooth plane rising 1 in 5 by a force equal to the weight of 200 lb. (*a*) when the force acts up the plane? (*b*) when it is horizontal?

(8) A heavy ball hangs from a point by a string. A second string is attached to it, and by this the ball is drawn aside so that the first string is no longer vertical. Draw a figure showing a triangle with its sides proportional to the three forces acting on the ball, and observe how they change with the inclination when the pulling string is kept horizontal.

(9) Draw a diagram to scale showing the resultant of two forces equal to the weights of 7 and 11 lb. acting on a particle, with an angle of 60° between them; and by measuring the resultant find its magnitude. Indicate two equal forces, at right angles to each other, which would be equivalent to the above two forces.

(10) Calculate the magnitude of the resultant of two forces, of 35 and 40 units respectively, acting at the same point and making with each other an angle of 120°.

(11) Six forces, 3, 4, 7, 10, 9, 5, act from the centre of a regular hexagon towards the angular points. Find the magnitude and position of their resultant.

(12) Find, to two decimal places, the resultant of two forces, 20 and 12, both acting from the corner of a square, the former along the diagonal, the latter along a side.

(13) A cross wind strong enough to exert a pressure of 10 lb. per square foot on an object placed normally to its direction acts against a sail of 500 square feet area inclined at 45° to it and to the boat. Find the effective component of the wind-pressure on the boat, neglecting leeway; (*a*) when the boat is stationary; (*b*) when the boat is travelling at 10 miles an hour.

The latter case is to be solved by measurement from a diagram.

(14) Draw a square ABCD, and take E the middle point of BC. Forces of 10, 15, 20, and 25 units act at A, from A to B, D to A, C to A, and A to E respectively. Find their resultant by measurement from a diagram.

(15) A piece of wire 26 inches long, and strong enough to support directly a weight of 100 lb., is attached to two points 24 inches apart in the same horizontal line. Find the maximum load that can be slung on the middle of the piece of wire without breaking it.

(16) The sides AB, AD of a rectangle ABCD are 5 and 12 inches long respectively. Forces of 8 and 20 lb. weight act at A in the direction AB and AC respectively. Find their resultant, either by construction or calculation.

(17) In what time will a body slide down 4 feet of a rough incline of 30° for which the coefficient of friction is $\frac{1}{10}$?

(18) What coefficient of friction will enable a weight just to rest on an inclined plane of 30° without extra support?

(19) Find the least force that will pull a hundredweight up such a plane as that in No. 18, or that in No. 17, and show that the best angle of traction is in general equal to the angle of repose.

COMPOSITION OF FORCES ACTING ON A RIGID BODY.

118. For the case of a rigid body, in addition to the *magnitude* and *direction* of the resultant as determined by the polygon construction, sect. 108, it is necessary also to determine its *position*—that is, its line of action. For observe that, though as regards translation a force in one place is as good as an equal parallel force in another, yet as

regards rotating power its position is important. Thus, imagine a long trough of water lying on the ground with a string tied to it by which you wish to raise it. Any vertical force greater than the weight of the trough must needs raise it, wherever the string is tied; but if the string is tied anywhere except above one definite point, the trough will also turn round as it rises, and the contents will be upset.

Again, if you raise it by two parallel strings, one near each end, then when the pull of the two strings together is a little greater than the weight of the trough, it is raised; but if you want to raise it without rotation, the pull of each string must be carefully proportioned, so that the resultant of the two forces may pass through the point above spoken of, which is called the centre of gravity.

Again, in the case of a pivoted body, it is obvious that a force applied close to the pivot has much less effect than an equal one far off; and if applied *at* the pivot, it can have no motive effect whatever.

119. Now, the fundamental dynamical idea in rotation is the *moment of a force* (read sect. 57 again); and the following general statements are true, with their converses.

(1) The moment of the resultant must equal the sum of the moments of the components about any point in every possible case, otherwise the resultant would not be truly the resultant, because unable to replace the components in rotating power.

That this condition is fulfilled by the diagonal of a parallelogram whose sides represent the component forces may be proved among other ways as follows:

To show that the resultant given by the parallelogram of forces is equivalent to its components in **rotating** ***as well as in translating power—that is, that its moment about any point in the plane is***

equal to the sum of the moments of the two components. The moment of the force AB about a point O (fig. 34) is (see sect. 53) geometrically representable by twice the area of the triangle OAB; the moment of AD is similarly proportional to twice the triangle OAD, and that of AC is twice OAC: hence what we have to prove is the following equality between the areas, OAB+OAC=OAD; the point O being in the plane of the parallelogram.

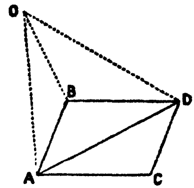


Fig. 34.

Now OAC=OBD+ADB, because the bases are equal, and the height of the single triangle is equal to the sum of the heights of the others (this is an easy extension of Euc. I. 38—analytically obvious, thus $\frac{1}{2}b(h_1+h_2)=\frac{1}{2}bh_1+\frac{1}{2}bh_2$);

and by inspection, OAD=OAB+OBD+ADB;

therefore OAD=OAB+OAC;

which was to be proved.

(2) The algebraic sum of the moments of any number of forces about a point on their resultant is zero; in other words, the sum of the positive moments equals the sum of the negative. (The moments of *two* forces about a point on their resultant are therefore numerically equal, but of opposite sign.)

For their resultant can have no rotating power about such a point, neither therefore can the components.

(3) If the body on which the forces act has one point fixed, and if their resultant passes through the fixed point or pivot, it will not be rotated by them.

For instance, to keep the beam (fig. 37) steady, C is the point to fix. The pressure of the pivot or fulcrum is then equal and opposite to the resultant of all the forces.

(4) The resultant of two forces acting on a rigid body passes necessarily through their point of intersection. If they do not intersect even when produced (which can only be by reason of their lying in different planes), then they have no resultant; and they cannot be further reduced. The effect of such an irreducible pair of forces is to carry a

body along and twist it round at the same time. They are not to be confused with a simple 'couple,' which results from the intersection of given forces at infinity: quite a different thing geometrically from not intersecting at all.

Composition of Two Forces in General.

120. If the two forces are in one plane, the parallelogram is a complete solution, whether they act on a particle or a rigid body, for the forces must intersect somewhere, and the point of intersection fixes the position of the resultant.

Thus fig. 35 is the physical part of fig. 29 repeated for a rigid body, say a stone pulled by two strings. The geometrical part applies just as well as before. The direction of the resultant must pass through E, the point where the given forces produced backwards intersect, and it may be applied to the body at any point in a line EP parallel to AD (fig. 29).

It may happen, however, that the point of intersection of the two forces is inconveniently distant, as off the paper for instance, or even at infinity when the forces are parallel. In such cases the general construction of sect. 123 is resorted to.

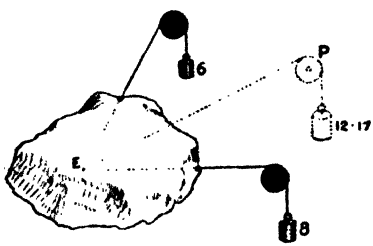


Fig. 35.

121. If the two forces are not in one plane they cannot intersect, and our construction for finding the resultant fails. The fact is *they have no resultant*, and cannot be further reduced; they can only be put into the more convenient form of a force and a 'couple' (sect. 128) in a plane perpendicular to the force; so they tend to carry the body along and turn it round at the same time. This pair of forces is called a *wrench*, because it tends to twist the body about a certain screw; but the subject now becomes too complicated for us in this stage. (This is what was meant in sect. 106 by the two forces to which any forces

whatever acting on a rigid body can always be reduced, even when no more can be done.) If, however, all the forces lie in one plane, no 'wrench' is possible, and they may then always be reduced to one simple resultant; though it may be a resultant zero at infinity as one special case, in which case it is most easily treated as 'a couple.'

Composition of any number of Forces in a Plane.

122. The parallelogram construction may be applied several times in succession, reducing the number of forces by one each time. This is a complete but cumbrous solution.

The polygon construction is a solution as regards magnitude and direction, but requires supplementing in order to determine position. The supplementary construction employed is such an important one, that it seems well to introduce it here, although its full discussion would lead us beyond our present mark. It will be best understood by an example, and the case of only three forces will afford a sufficient illustration of the method. It depends on the fact that a single force may be resolved into a pair of components in an infinite variety of ways (fig. 31); so that, if the given forces are not convenient to find the resultant from, we can choose a more convenient pair out of the set which have the same resultant, and then draw the resultant of these. Expressed in another way, it may be said to depend on the fact that forces in equilibrium produce no disturbance, and hence may be introduced or removed at pleasure.

Construction for finding the line of action of the Resultant of any number of Forces whose directions all lie in one Plane.

(Illustrated by the case of three forces.)

123. Let P, Q, S (fig. 36) be the forces. Draw the sides of the polygon ABCD parallel to, equal to, and in the

same sense as the three forces; then the completion of the polygon, AD, is the resultant R in magnitude and direction. Where is it to be placed?

Choose *any* point O, join OB, and draw in the other diagram a line PQ parallel to it across the forces P and Q (the line is to be drawn across P and Q, because B is the

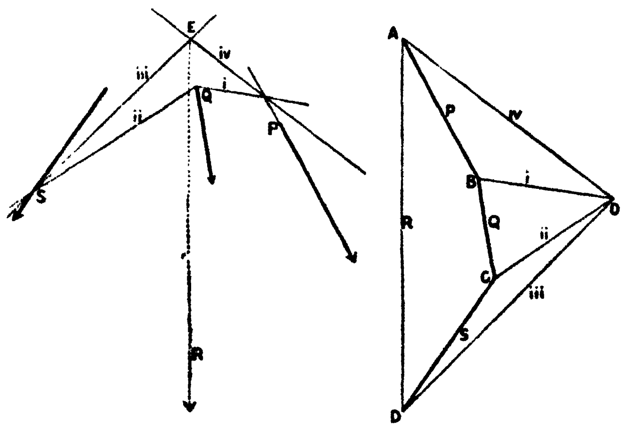


Fig. 36.

[Lines parallel to each other in the two diagrams are labelled similarly; but the simplest way in practice of indicating the correspondence of lines is to mark the areas in one figure and the points in the other. Thus, for instance, the letters P, Q, R, S, in the right-hand figure, may be supposed to denote the areas containing these letters, and the letter E may be affixed to all the external space. In the left-hand figure these same letters belong to points, especially if R be understood to apply to all the arrow-heads; and all lines in either figure are denoted by two letters, in the one by points, in the other by areas which they separate. Thus the line QS is the line joining (or separating) the points Q and S in one figure, and is the line separating the space Q from the space S in the other.]

meeting-point of the sides of the polygon which represent P and Q).

Then join OC, and draw a line QS parallel to it (C being the meeting-point of the sides representing Q and S).

Then join OD, and draw a line through S parallel to it,

say SE; also join OA, and draw a line through P parallel to it.

The point E, where these last two lines intersect, will be a point on the resultant, and its position is therefore determined. *Q.E.F.*

Proof (This may be omitted till after Chapter VIII. has been read).—A force represented by AB is the resultant of two forces represented by AO and OB, because it forms a triangle with them; but P is a force represented by AB, hence P is equivalent to two forces acting along the lines EP and PQ, equal to AO and OB respectively, and may be replaced by them. Let it be so replaced. Similarly the force S may be replaced by two forces acting along SQ and ES, equal to CO and OD respectively.

But Q is in equilibrium with two of these forces—namely, those along SQ and PQ, since their representative lines, BC, CO, OB form the sides of a triangle taken in order (sect. 135); hence this set of three forces may be removed; and there now remain, as the equivalents of the original forces, P, S, and Q, only forces along EP and ES, represented by AO and OD respectively. Hence these two forces have the same resultant as the three original forces had; but the resultant of these two forces passes through E, their point of intersection (sect. 120); therefore the resultant of the three original forces, P, Q, and S, passes through the point E. *Q.E.D.*

It will be seen, therefore, that what we really do in the construction, is to compound with P, Q, and S, two sets of equilibrating forces—namely, two equal opposite forces in the line PQ of magnitudes OB and BO, and two in the line QS of magnitudes OC and CO; and by their help to replace the given forces by two intersecting ones, ES and EP, the position of whose resultant is obvious.

This construction applies equally well to parallel forces, only then, of course, the polygon ABCD shuts up, the points B and C being on the straight line AD; but everything else remains without modification.

The use of the above construction may not be quite

apparent perhaps, but it is put here as an indication of quite a large art—namely, *graphical statics*—which may well occupy the student's attention at a later stage. The quadrilateral ABCD on the right of fig. 36 is called 'the force-polygon,' and determines the magnitude and direction of the resultant; the quadrilateral i, ii, iii, iv on the left of fig. 36 is called 'the funicular polygon,' and determines its position. It is called the funicular polygon or sometimes the link polygon, because it represents the equilibrium directions of a weightless string or linkage of rods subject to the given forces. An arch or framework composed of jointed bits of wood, two of them placed like i and ii and two of them like iii and iv produced, would be in equilibrium under the given forces PQS if its terminal ends were fixed. Or if the forces were reversed in direction, it would represent the shape taken by a string with fixed ends subject to the given forces. Another but less obvious statement which can be made is that the polygon i ii iii iv in the left-hand figure represents the distribution of bending-moment in a beam subjected to the force R, and to the forces PQR reversed; it should have a thickness graduated according to this manner if it is to be as stiff as possible without waste of material.

Composition of Parallel Forces.

124. Parallel forces can only act on an extended body: forces which act on a particle, of course, cannot be parallel. The direction of the resultant of parallel forces is the same as the common direction of its components, while its magnitude is their algebraic sum—that is, their sum paying regard to sign—adding all that act in one direction, subtracting any that pull the other way. This is all that is required to be known for translation (sect. 118); but to discuss the *rotation* of a body under the influence of parallel forces, we must learn the *position* of the resultant, and this requires

either a geometrical construction or an arithmetical calculation.

The general construction of sect. 123 applies to parallel just as well as to other forces, so we have only to give the method of calculating its position arithmetically.

125. The fact (No. 1, sect. 119) that the moment of the resultant equals the algebraic sum of the moments of all the components, though universally true, is most useful in its application to *parallel* forces, and it affords a ready method of finding the position of their resultant arithmetically.

Thus imagine a weightless beam acted on by any parallel forces, say weights, 4, −5, 6, −2, &c., arranged anywhere on the

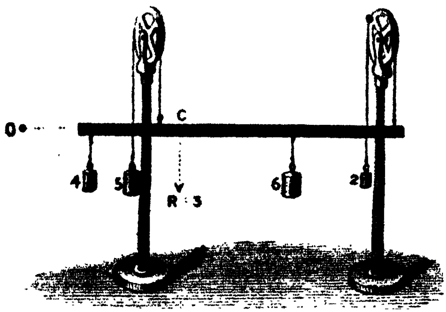


Fig. 37.

beam (as shown in fig. 37), at distances 4, 8, 16, 22 inches from some fixed point of reference O; then the resultant R is, in magnitude,

$$4-5+6-2=3,$$

and is at a distance x from O, such that

$$3x=4\times4-5\times8+6\times16-2\times22=28;$$

wherefore $x=9\frac{1}{3}$ inches. Mark off OC equal to this; then R acts at the point C, as shown; and, to keep the bar in equilibrium, another pulley and string must be arranged to exert a force 3 upwards at this point.

And, generally, if the forces be w_1, w_2, w_3 ...... at respective perpendicular distances x_1, x_2, x_3 ...... from any point O, then the distance of the resultant from the same point is

$$x = \frac{w_1 x_1 + w_2 x_2 + w_3 x_3 + \ \ldots\ldots}{w_1 + w_2 + w_3 + \ \ldots\ldots}.$$

This is a constantly occurring form of fraction, and is a more general sort of average. If $w_1 = w_2 = w_3 =$ &c., then it would be the ordinary expression for finding the average of the distances x_1, x_2, x_3, &c.—that is, it would give the average distance of all the weights from O; for it would add all the distances together and divide by the number of them.

To take another example, let weights of 8, 6, 4, 2 pounds respectively be hung at the following inch divisions of a one-foot rule: 0, 3, 9, 12. Find the position of the resultant. Let it be at the division x, and take moments about one end. The moments of the weights are respectively 0, 18, 36, and 24; the moment of their resultant is $20x$; wherefore, equating this to the sum of the separate moments, we get $x = 3{\cdot}9$.

Once more, let there be only two weights, say 4 lb. at one end of a rod a foot long, and 8 lb. at the other. Then calling the distance of the resultant from the smaller weight x, its distance from the bigger weight is $12 - x$, and taking moments about the resultant, we have $4x = 8(12 - x)$, whence $x = 8$ inches. That is, it divides the rod in inverse ratio to the two weights, and this is a general result.

Composition of Two Parallel Forces.

126. When we have only two forces to deal with, the general statements and constructions are of course equally applicable, but they may be put into a more simple form. The resultant is equal to the sum of the forces if they act

in the same direction, and is equal to the difference of the forces if they act in opposite directions; it is of course parallel to either force, and it only remains to find its position. The following three simple constructions may be given for finding the position of the resultant geometrically.

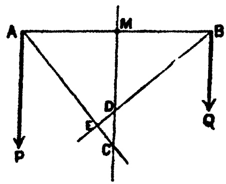


Fig. 38.

First Construction (fig. 38).—Take a point M half-way between the forces P and Q, and draw two lines through it; one parallel to the forces, the other not, but cutting them in A and B respectively. Lay off from M two lengths in the former of these lines, in the same sense as the respective forces, MC equal to P, and MD equal to Q, and join AC and BD; the resultant shall pass through E, the intersection of AC and BD.

Proof.—We may suppose that we have here compounded P with a force AM, and Q with an equal, opposite, and therefore equilibrating, force BM; and AC, BD are the diagonals of parallelograms, and have the same resultant as P and Q have.

Second Construction (fig. 39).—Anywhere on the line of P take a length equal to Q, and on the line of Q a length equal to P. Then draw straight lines joining the extremities of these two lengths; they will intersect in a point on the resultant, and so determine its line of action.

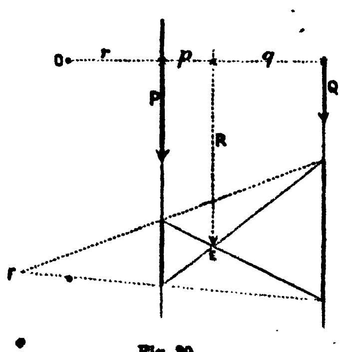


Fig. 39.

If the forces have the same sense, they are to be joined crosswise, and E is the point. If they have contrary sense, they are to be joined without crossing, and F is the point.

Proof.—Observe that the two triangles with common vertex E (or F) are similar (their bases being parallel), hence their heights and bases are proportional. But their bases are Q and P; so, calling their heights p and q,

$$\frac{p}{q}=\frac{Q}{P},$$

or

$$Pp=Qq.$$

This fact proves the proposition; for, by (2) sect. 119, the moments Pp and Qq of the two forces about their resultant must be equal and opposite; but they are evidently *opposite*, from the figure, and this equation states their *equality*, about the point E, and similarly about the point F. Wherefore the resultant passes through E if the forces have the same sense; and through F if they have contrary sense.

It is worth noticing that the moment of R about *any* point O is equal to the sum of the moments of its components; for, taking the circumstances as depicted in fig. 39, we can easily show that $R(p+r)=Pr+Q(p+q+r)$, by remembering that $R=P+Q$, and that $Pp=Qq$.

Third Construction (fig. 40).—A more general construction, applicable whether the forces are parallel or not,

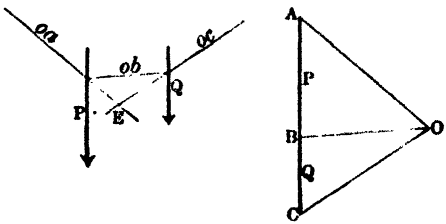


Fig. 40.

and practically useful when the forces are nearly parallel so that their point of intersection is inconveniently far off, is based upon the method of the funicular polygon (sect. 123).

Draw a line AB representing the force P, and a line BC representing Q. Choose any point O, and join OA, OB, OC. Then draw across the given forces a line parallel

to OB, and at its points of intersection with P and Q respectively draw other lines parallel to OA and OC. The point E where these last two lines meet is a point on the resultant, and determines its line of action; its magnitude and direction are given by AC.

Proof.—The force P has been virtually replaced by forces AO, OB acting along the lines *ao, ob*; the force Q has been virtually replaced by forces BO, OC acting along the lines *bo, oc*. Hence the resultant force is replaceable by AO, OC in the lines *ao, oc*, and therefore passes through their point of intersection.

127. The following propositions concerning two parallel forces are now at once seen to be true, being little more than repetitions in a compact form of what has gone before.

(1) The distances between each force and the resultant are inversely as the forces—that is, $p : q = Q : P$. This formula, or some other *moment* formula, must be used when you want to find the resultant arithmetically.

(2) If two parallel forces have the same sense, their resultant is equal to their sum, and lies between them, nearer the bigger one. In fig. 39 it passes through E. If, however, they are of contrary sense, their resultant equals their difference, and lies outside them on the side of the bigger one, agreeing with the bigger one in direction. In fig. 39, if one of the given forces is reversed in direction, the resultant passes through F.

(3) If two forces are equal, the resultant must be equidistant from both.

If they are of contrary sense, this means that the resultant is at infinity; but its magnitude is zero, being equal to the difference of the components.

128. Hence, two equal contrary* parallel forces have a

* The phrase 'non-concurrent' has been used to express parallel opposition not in the same line, but the word *contrary* is to be preferred, since the proper meaning of the word 'concurrent' is *meeting in a point*.

resultant zero at infinity; or, as it is sometimes expressed, they have no resultant at all. (In any of the constructions the lines whose intersection gives the position of the resultant will for this case be found to be parallel.) Such a pair of forces cannot be further simplified, hence they are taken together and called a *couple*. The moment of the couple about any point will be easily seen to be independent of the position of that point, and to equal either force multiplied by the perpendicular distance between the two forces, this distance being called the *arm* of the couple.

A couple is not properly to be regarded as *two* forces, but as a particular case of *one*—namely, an infinitely small force at an infinitely great distance. It obviously possesses only *rotating* power. The fact that its moment about every point in its plane is the same causes its position to be unimportant. Its moment and its plane have to be specified, but nothing else. (Read again sect. 106.)

The Composition of Parallel Forces as illustrated by Gravity.

(Centre of Gravity.)

129. The force of gravity illustrates the subject of parallel forces very well. A rigid body is made up of particles, every one of which is pulled towards the centre of the earth with a force proportional to its mass, and equal to its mass multiplied by g (sect. 64). Now, since the centre of the earth is such a long way off, these converging forces are for bodies of ordinary size practically parallel. Hence the whole pull of gravity on a table or a book is really the resultant of an infinite number of parallel forces—the attractions on the several particles.

To find the *magnitude* of this resultant, you hang up the body on a spring balance—in ordinary language, you *weigh* it.

To find its *position*, the easiest way is to hang up the

body by a bit of string; the line of the resultant is then a continuation of the string, since it must pass through the point of suspension. Or you may balance the body on your finger; the line of the resultant is always the vertical through the point of support whenever the body is in equilibrium.

Its *direction* is a fixed one—namely, always pointing to the centre of the earth, no matter how you turn the body.

Now when a rigid body exposed to the action of a number of parallel forces acting at definite points in the body is turned about, there is one point in the body through which their resultant always passes in every position—and this point is called the *centre of the parallel forces*; or, if the parallel forces are due to gravity, it is called the *centre of gravity*.

The criterion as to whether there really is such a point in general is rather troublesome; and if the forces are not accurately parallel, there is, strictly speaking, no such point for bodies of irregular shape. Nevertheless, the forces due to gravity acting on the parts of any body of reasonable size are so nearly parallel that practically everything likely to be experimented on has a centre of gravity.

Determination of the Centre of Gravity by Experiment.

130. If this point be directly supported, the body is in equilibrium in *every* position necessarily; and conversely, if a body is in such equilibrium, it must be because its centre of gravity is directly supported. (A coach-wheel, for instance, should be pivoted at this point.) Hence this gives one way of finding it. Another way of experimentally determining its position is to find out the line of the resultant in some two positions of the body by hanging it up twice in different ways (see fig. 41); then the centre of gravity must be the point common to the two lines—

that is, it must be where they cross. However the body be hung up by a single point, the centre of gravity will always, when at rest, be vertically under or over the point

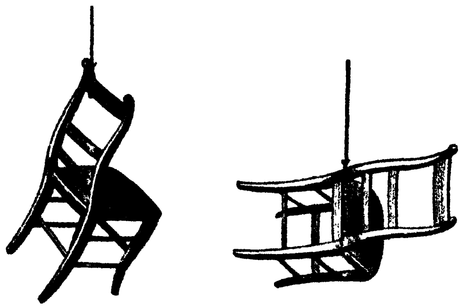


Fig. 41.

of suspension; that is, the line of the resultant will always pass through the fixed point.

The whole weight of a body, then, may be considered to act at its centre of gravity; in other words, it behaves *for purely statical purposes* as if the whole mass of the body were concentrated at this point.

Determination of the Centre of Gravity by Calculation.

131. The centre of gravity is always the most symmetrical point in a body. In a sphere it is the centre; so it is also in a cube or an ellipsoid, and in a square or circular plate. In a parallelogram or a parallelopiped (that is, solid parallelogram), it is the intersection of diagonals. In a rod of uniform thickness and material, it is the middle; and so on.

But it is easy to calculate its position in less uniform cases by any process which will determine the position of the resultant of a number of parallel forces, for it is simply the point through which the resultant *always* passes.

In fact, if m_1, m_2 ...... are the masses, at the distances

$x_1, x_2, x_3, \ldots\ldots$ from any line in the plane (restricting ourselves to masses distributed in a plane), the distance x of their centre of gravity from the same line is given by

$$x = \frac{m_1x_1 + m_2x_2 + \ldots\ldots}{m_1 + m_2 + \ldots\ldots} \quad \text{(see sect. 125).}$$

And if the distances of the centre of gravity from two such lines (not parallel) are found, its position is completely determined. In some cases, one line on which the centre of gravity must lie is obvious, and then all that is further necessary is to determine its distance from any given point on that line, as in the following example.

Thus let this rod with middle point M (fig. 42) be of weight two pounds, and let a ring A, weighing half a pound, be placed on it four inches to the left of M, and another ring B, weighing three pounds, six inches to the right; then, if the rings fit tightly, the centre of gravity of the whole must lie somewhere in the length of the rod. To find whereabouts, we need only calculate the position of the resultant of the three weights (the two rings and the rod itself) in any position except the vertical one, say when horizontal. The magnitude of the resultant is plainly $5\frac{1}{2}$. Take moments about any point, say about A; let the resultant act at some unknown point C, such that AC $= x$. Then we have $5\frac{1}{2}x = (3 \times 10) + (2 \times 4) + (\frac{1}{2} \times 0) = 38$; wherefore $x = \frac{76}{11} = 6\frac{10}{11}$; or the point C is $2\frac{10}{11}$ inches to the right of M, and it is the centre of gravity. If the bar is to be supported in horizontal equilibrium, it must be pivoted or suspended by this point, or, better, by a point just above it (see sect. 145).

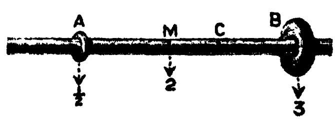


Fig. 42.

Try now taking moments about M, also about B, also about O (anywhere), and see that you always get the same result (when interpreted properly), remembering to allow for negative moments.

In this example it has happened, as stated above would often happen, that the line of the resultant in one position of the body (in this case when the rod is vertical) is perfectly obvious.

The arithmetical determination of the position of the

centre of gravity of a body, therefore, depends on precisely the same principle as the experimental method, and consists simply in finding the line of the resultant in any two positions of the body, and noting their point of intersection. It therefore scarcely needs further exposition; but it is probably necessary to show how this same principle is applicable to cases rather less obvious.

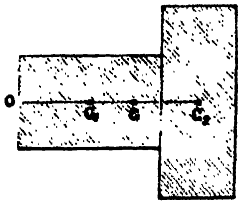


Fig. 43.

For instance, to find the centre of gravity of a body made of two parts, each part having a known centre of gravity; say two flat oblong plates, of known weights, w_1 and w_2, joined end to end. G_1 and G_2 being the centre of each separately, their weights may be considered as acting here, and so the resultant passes through a point G, which divides the line G_1G_2 in the ratio $w_2 : w_1$ (sect. 127)—that is, so that $G_1G : GG_2 :: w_2 : w_1$; hence G is the centre of gravity of this combination.

Or we might take moments about any point O, and say

$$w_1 \cdot OG_1 + w_2 \cdot OG_2 = (w_1 + w_2)OG,$$

whence the distance OG, and therefore the position of G, is determined.

The same method applies if a bit is taken away instead of added on. Suppose, for instance, a square plate with a round hole in it anywhere (fig. 44). The operation of finding the centre of gravity in such a case may be regarded as the same as that of finding the position of the resultant of two contrary forces—the weight W of the whole square acting downwards at G_1 and the weight w of the missing bit acting *upwards* at G_2. The centre of gravity G must evidently be somewhere in the line G_1G_2; so, taking moments about any point O in this line, the equation $W \cdot OG_1 - w \cdot OG_2 = (W - w)OG$ determines its position.

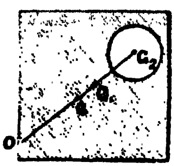


Fig. 44.

Or more simply thus: let x be the distance of G from the centre of the square G_1, and let a be the distance of the centre of the hole G_2 from the same point G_1, then write $Wx = w(a + x)$ and solve for x.

Or again, the centre of gravity of a trapezium (that is, a quadrilateral with two parallel sides), which may be regarded as a triangle with the top missing, can be found in precisely the same way. The last equation applies as it stands, in fact, provided we know the positions of G_1 and G_2, the centres of gravity of the whole and of the missing triangles (see fig. 45).

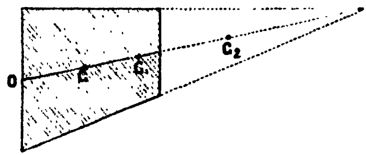


Fig. 45.

The centre of gravity of a triangular plate is in the line joining a vertex to the bisection of the opposite base (because this line bisects every line in the triangle parallel to the base). Three such lines can be drawn, because there are three vertices. Therefore these three lines, joining each vertex to the middle point of the opposite side, meet in a point, and that point is the centre of gravity. It is easily seen to divide each line in the ratio 1 : 2—that is, it is one-third of the way up from the base to the vertex.

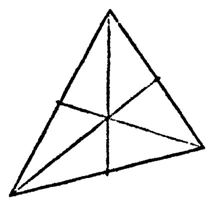


Fig. 46.

The centre of gravity of any quadrilateral can be found by dividing it into a pair of triangles in two different ways, and taking the crossing-point of the lines joining the triangles' centres of gravity.

The following statements may for the present be assumed for the sake of examples.

The centre of gravity of any pyramid or cone is in the line joining the vertex to the centre of gravity of the base

and one quarter of the way up. The centre of gravity of a solid hemisphere is $\frac{3}{8}$ths of its radius from its flat boundary —that is, from its geometrical centre; of a hollow hemisphere is half-way between centre and circumference; of a semicircular area is $\frac{4r}{3\pi}$, or approximately $\frac{3}{7}$ of its radius from the centre; of a semicircular arc (like a wire) is $\frac{2r}{\pi}$, or approximately $\frac{7}{11}$ of its radius from the centre.

EXAMPLES—XVII.

(1) Resolve a force 20 into two parallel forces, one of them 3 times as far from the given force as the other.

(2) A weightless curtain rod has 4 equal rings on it, so that the 2 end rings are 5 feet apart, and the 2 middle rings are 1 foot apart, one of the end rings being 18 inches from the nearest middle one. Find the centre of gravity.

(3) Where would the centre of gravity in the last question be, if the rod itself were 5 feet long, and weighed twice as much as a ring?

(4) A uniform circular disc has a circular hole punched out of it, extending from the circumference half-way to the centre. Find the centre of gravity of the remainder.

(5) Forces of 1, 2, 3, and 4 lb. weight act along the sides of a square whose diagonal measures 4 inches. Find the magnitude and position of their resultant.

(6) Prove that the moment of a given 'couple' is the same about every point in its plane—that is, that a 'couple' has magnitude and direction but no position. What is the moment of a couple consisting of two equal contrary parallel forces, of 5 lb. weight each, separated by a perpendicular distance of 12 inches from each other?

(7) A uniform beam 10 feet long, weighing 80 lb., is suspended from two points in a horizontal ceiling, 16 feet apart, by strings each 5 feet long attached to its ends. Find the tension in each string.

(8) An iron sphere weighing 50 lb. is resting against a smooth vertical wall and a smooth inclined plane which is inclined at 60° to the horizon. Find the pressures on the wall and plane.

(9) Find the resultant of parallel forces 1, -2, 4, -3, acting at equal distances, of one foot each, along a weightless beam, the negative sign indicating that the force acts upward.

(10) Three forces represented in magnitude, direction, and position by OA, OB, and OC, are in equilibrium. Show that

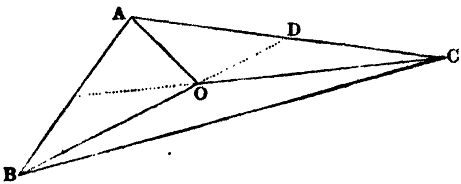


Fig. 47.

O is the centre of area (the so-called centre of gravity) of the triangle ABC.

Solution.—	OA	is equivalent to	OB and BA.
and	OC	"	OB and BC.
∴	BO	"	2OB and BA and BC.
∴	3BO	"	BA and BC.
But	BA and BC	"	2BD if D bisects AC.
∴	3BO equals 2BD in magnitude and direction.		

Q.E.D.

(11) A weight of 100 lb. is fixed to the top of a weightless rod or strut 5 feet long, whose lower end rests in a corner between a floor and a vertical wall, while its upper end is attached to the wall by a horizontal wire 4 feet long. Calculate the tension in the wire, and the thrust in the rod.

(12) A uniform rod 8 feet long, weighing 18 lb., is fastened at one end to a vertical wall by a smooth hinge, and is free to move in a vertical plane perpendicular to the wall. It is kept horizontal by a string 10 feet long, attached to its free end and to a point in the wall. Find the tension in the string, and the pressure on the hinge.

(13) A uniform ladder 20 feet long, weighing 60 lb., is supported horizontally by two men at distances of 4 and 5 feet respectively from its ends. Find the weight borne by each man.

(14) A ladder 20 feet long, whose centre of gravity is 8 feet from one end, is carried horizontally by two men, who each carry the same weight. If one of them is at the heavier end, how far must the other be from the light end?

(15) A uniform beam, 24 feet long and weighing 200 lb., is supported on two props, one 6 feet from one end, the other 9 feet from the other end of the beam. Calculate the pressure on each prop when a man weighing 180 lb. stands on the beam 8 feet from the first prop.

(16) From one corner of a square plate, whose side is 10 inches, a small square, whose side is 3 inches, is cut away. Find the centre of gravity of the remainder.

(17) A circular hole 2 inches in diameter is cut in a uniform circular plate six inches in diameter, the centres being one inch apart. Find the centre of gravity of the perforated disc.

(18) Find the centre of gravity of weights of 7, 6, 9, and 2 lb., arranged at the corners of a square of 1 foot side.

(19) Weights of 1, 3, 5, and 7 lb. are placed at the corners of a uniform square plate of 10 inches side, weighing 4 lb. Find the centre of gravity of the system.

(20) Find the centre of gravity of weights 2, 3, 4, 5, 6, and 7 lb. placed at the corners of a regular hexagon whose diagonally opposite corners are 18 inches apart.

(21) Show that the centre of gravity of a uniform triangular plate coincides with that of three equal masses placed (*a*) at its angular points; (*b*) at the middle points of its sides.

(22) A triangular plate weighing 5 lb. is in shape an isosceles triangle with its two equal sides each 5 feet long, and its base 8 feet long. A weight of 10 lb. is hung at its vertex. Find the centre of gravity of the whole.

(23) The mass of a plate in the form of an equilateral triangle 1 foot high is 4 lb. Masses of 1, 1, and 2 lb. respectively are placed at its angular points. Find the centre of mass of the system.

(24) From a square plate, whose diagonal is 21 inches, a corner is cut off by a line joining the middle points of two adjacent sides. How far from the centre of the square is the centre of gravity of the remainder?

(25) If, in the previous question, the corner, instead of being cut

off, is folded over flat on to the plate, find the centre of gravity of the whole.

(26) Weights each equal to 1 lb. are placed at the ends of one side of a uniform plate 12 inches square weighing 2 lb. Determine the centre of gravity of the system.

(27) Two similar uniform bars, 4 and 6 feet long, are joined end to end at right angles so as to form an L. Find the centre of gravity of the system.

(28) A uniform iron window-frame, 2 feet high, is in shape three sides of a square, surmounted by a semicircular arc. Find its centre of gravity.

(29) A uniform iron plate, 2 feet high, is in shape a square surmounted by a semicircle whose base coincides with a side of the square. Find its centre of gravity.

(30) A circular portion 5 inches in radius is removed from a circular lamina 8 inches in radius, the distance between the centres of the two circles being 2 inches. Find the centre of gravity of the remainder.

(31) If the original plate was wood, and if the hole is filled up with lead of the same thickness, but 12 times as heavy bulk for bulk, where is the centre of gravity?

(32) Find the centre of gravity of a uniform plate 8 inches square containing a circular hole of 2 inches diameter, the centre of the hole being 2 inches from the centre of the plate.

(33) The middle points of opposite sides of a rectangular plate being joined, one of the four parts of the rectangle is removed. By what fraction of the diagonal is the centre of gravity of the remainder distant from the centre of the rectangle?

(34) Find the centre of gravity of a uniform quadrilateral plate whose sides are 6, 4, 3, 4 inches long respectively, and whose two equal sides are equally inclined to the others.

(35) Find the centre of gravity of a frame made of uniform bars arranged as above.

(36) Where is the centre of gravity of a slate-frame, 9 inches by 12, of which one of the short bars has been removed?

(37) Where is the centre of gravity of a triangular frame of sides 5, 4, 3? Where is that of a triangular plate of same size?

CHAPTER VIII.

ON EQUILIBRIUM (*Statics*).

132. Before leaving the subject of motion as affected by force, there is one important part to be considered—namely, the conditions under which forces may act on a body without affecting its motion in any way whatever. One force cannot satisfy these conditions, but a combination of any number of forces greater than one may; and it is interesting, and for many practical purposes important, to be able to specify these conditions, and to decide in any given case whether they are satisfied or not. This part of the subject is called 'Statics,' and it is a branch of the more general science of Dynamics. Its treatment will depend upon the ideas illustrated at length in the last chapter, which may be regarded as an introduction to Statics; indeed, they are usually considered as a part of it, and often are made to follow, or are mixed up with, the subject of the present chapter.

133. When all the forces applied to any mass of matter are so balanced that they produce no acceleration in it of any kind, the forces are (or the body is) said to be *in equilibrium*, and the conditions which they then necessarily satisfy are called the *conditions of equilibrium*.

Observe that equilibrium does not mean *rest* or zero *velocity*, it simply means zero *acceleration*—that is, constant velocity. There is no occasion for the velocity to be nothing; all that is meant is that it keeps the same value, whatever that may happen to be. Thus in the case of a bucket lowered down a well, suppose that it is descending with a constant velocity of 20 feet a second; then, its acceleration being zero, the resultant force acting on it

(being equal to mass-acceleration, sect. 45) must also be zero. Now the actual forces acting on it are the pull of the earth downwards, and the pull of the rope upwards; and the resultant of these two being zero, it follows that they are equal. The bucket is in fact simply obeying the first law of motion. Whether it is descending or ascending or standing still, matters nothing, the tension in the rope is always equal to the weight of the bucket so long as its velocity is not *changing.* The conditions of equilibrium are therefore the conditions under which acceleration is impossible; or, as it is often correctly expressed, they are the conditions under which *rest* is *possible.* It must be observed that forces in equilibrium have no influence in *causing* rest. They have no effect on the motion at all, and the body exposed to such forces simply obeys the first law of motion. Rest is zero velocity. Equilibrium is zero acceleration.

134. This being clear, we will proceed to state the conditions of equilibrium for any number of forces, and first of all

The Conditions of Equilibrium for Two Forces.

The conditions which two forces have to satisfy in order to balance each other and have no effect on the motion of the body to which they are applied, are very simple and obvious—namely: (1) The forces must both lie in the same straight line; (2) They must act in opposite directions; and (3) They must be equal.

This is all usually expressed by saying simply that the two forces must be *equal and opposite*, the acting in the same straight line being understood.

If any number of forces are in equilibrium, the resultant of any number of them must be equal and opposite to the resultant of all the rest. For obviously all the rest are equivalent to their resultant, and that resultant is balanced

by a force equal and opposite to it. The statement just made is not to be quoted as a condition of equilibrium, it is merely a manifest fact which may help us to ascertain the conditions of equilibrium.

135. Let us see how it gives us the equilibrium conditions for *three* forces, for instance. Any one force must be equal and opposite to the resultant of the other two. Now any two of them, as A and B, in order to have a resultant, must lie in one plane, in other words, must (if produced) meet in a point, and through this point their resultant must pass, being the diagonal of the parallelogram of forces; the third force, C, in order to maintain equilibrium, must, by the above statement, be a prolongation of this diagonal, and hence it too passes through the same point as the other two, and is in the same plane—namely, the plane of the parallelogram; it must also be equal to the diagonal in magnitude; in other words, it must be equal to the third side of a triangle, two of whose sides represent the other forces, such as OAR (fig. 48). Its magnitude, direction, and position are thus completely determined.

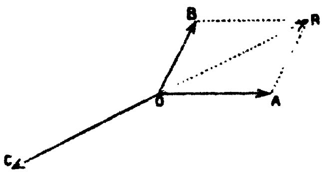


Fig. 48.

Let us restate these:

The Conditions of Equilibrium for Three Forces.

(1) The three forces must all be in the same plane.

(2) Their lines of action must all pass through the same point.

(3) It must be possible to draw a triangle with sides parallel (or perpendicular; see foot-note, sect. 111) to the forces, and proportional to them in magnitude. The sides of the triangle must all be drawn in the same *sense* as the forces (thus in the figure, OA, AR, RO are the senses), and

it must be possible to draw the triangle without taking the pen off. This is usually expressed by saying that the three forces must be representable by the sides of a triangle *taken in order*. The last two conditions together really include the first.

Any number of forces greater than three need neither meet in a point, nor lie in the same plane, in order to be in equilibrium.

Conditions of Equilibrium of a Particle.

136. Any number of forces acting on a particle will evidently be in equilibrium if they are representable by the sides of a closed polygon (plane or otherwise) drawn parallel to the respective forces and taken in order.

This is the same as saying that the forces must have no resultant; for the line required to complete the polygon represents the resultant (sect. 108), but no line is required to complete a *closed* polygon, hence there is no resultant.

The converse is also true—namely, that if forces acting on a particle are in equilibrium, they must be representable by the sides of a polygon taken in order. This proposition obviously includes the triangle of forces, for a triangle is only a three-sided polygon.

Conditions of Equilibrium of a Rigid Body.

137. If the condition just stated for a particle is satisfied by the forces acting on a rigid body, they can produce no translation, only rotation; hence a rigid body will evidently be in equilibrium if the above condition for a particle be satisfied, and if also the directions of all the forces pass through a single point; for a set of forces which intersect in one point, and have zero resultant, cannot possibly rotate anything. But this last condition, though *sufficient*, is not *necessary*—that is, the converse is not true: if the forces

acting on a rigid body are in equilibrium, they must indeed be representable by the sides of some closed polygon (plane or otherwise), but they need not meet in a point, unless there are only three of them. The more general condition for no rotation is that the moments of all the forces about every possible point or axis of rotation must add up to zero.

If this and the particle condition are satisfied, equilibrium is complete; and conversely, wherever there is equilibrium, these must be satisfied. So these are the necessary and sufficient conditions, though not in a very simple form to apply practically. It will be sufficient for us, however, to consider at length, and put into a more practical form, only the case where all the forces act *in one plane*; and we will proceed to this from a fresh point of view in the next section; but we may first notice a mode of expressing the conditions of equilibrium in terms of the construction of sect. 123.

If the force polygon is closed there is no resultant force, but there may be a resultant couple, causing rotation. In that case the funicular polygon cannot be closed by producing its first and last sides, for they will be parallel. Whenever the funicular polygon is closed there is no resultant moment, and the funicular polygon cannot be closed unless the force polygon is closed; so in cases of complete equilibrium, the force polygon and the completed funicular polygon are both closed.

General Conditions of Equilibrium of a Rigid Body acted on by Forces in a Plane.

138. The motions possible to a rigid body are translation or rotation or both, hence the conditions for equilibrium really involve the conditions for no translation and for no rotation (strictly speaking, for no rectilinear and for no angular *acceleration*; but the words translation and rotation

are used instead of these more accurate terms for shortness; and the error is not great, for the conditions of equilibrium render entire rest *possible*, though they do not in any way enforce it).

Now, having assumed that the body can only move in a plane (say a vertical plane), and that the forces only act in this plane, it is obvious that all translations must be up or down, or right or left, or else a motion compounded of the two, which may be analysed into up or down and right or left components. Hence, in order that there may be no translation at all, the forces must have no resultant either up or down or right or left: this being a practically convenient form of saying that they have no resultant at all at a finite distance. Still, however, they might spin the body (sect. 128); hence, in addition to the above, the condition is necessary that the sum of their moments about any point in the plane must vanish; and then the forces will be unable to cause any motion at all. Or the complete condition for equilibrium might be stated by saying that the sum of the force-moments about *every* point in the plane must be zero; since this necessitates the non-existence of either a resultant force or a resultant couple. This condition is, however, not so practically applicable, in general, as the two separate conditions just laid down, which we now repeat with emphasis:

The general conditions of equilibrium for a body only able to move in a plane are:

(1) That the sum of the components of all the forces in any two directions in the plane at right angles to each other shall vanish.

(2) That the sum of the moments of all the forces about any one* point in the plane shall vanish.

* *One* point is sufficient if condition 1 is satisfied, because the moment of a couple about every point is the same (sect. 128); hence, if it is zero about any one point, it is zero altogether.

(1) is the condition for no translation (properly speaking, for no rectilinear acceleration).

(2) is the condition for no rotation (properly speaking, for no angular acceleration).

If (1) is satisfied without (2), there is rotation, but no translation.

If (2) is satisfied without (1), there would be translation, but no rotation about the particular point considered. If (2) is satisfied for *every* point, then (1) is also necessarily satisfied, because the moment of an unbalanced force must differ for different points.

If neither is satisfied, there must be both translation and rotation.

If both are satisfied, there must be complete equilibrium.

The converse of each of these statements is also true.

In case the body on which the forces act has one point fixed so as to be incapable of translation, the necessary and sufficient condition for equilibrium is simply that the resultant of all the forces must pass through the fixed point or pivot (see sect. 119, statement 3). And in general, instead of applying both conditions (1) and (2), it would be sufficient to apply condition (2) to three different points not in the same straight line, but it would be more troublesome in practice.

ILLUSTRATIONS.

139. Consider a ladder standing on rough ground, and resting against a perfectly smooth wall. What forces are acting upon it? There is the weight of the ladder W acting downwards at its centre of gravity G (fig. 49); there is the pressure of the ground R acting in some unknown upward direction at some angle with the vertical not greater than the 'angle of repose' (sect. 115), and the pressure of the wall P acting normal to the wall or horizontally; and that is all. But the ladder is in equi-

librium, hence these three forces must pass through a point (sect. 135).

Now W and P, whose directions are known, intersect when produced in the point C; hence R also passes through the point C (fig. 50).

This determines its direction.

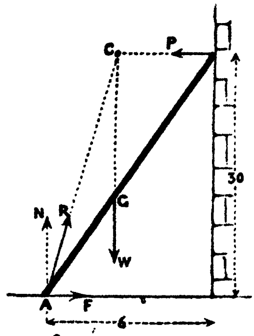


Fig. 49.

Moreover, when three forces are in equilibrium, they must be proportional to the sides of any triangle which are drawn respectively parallel to the forces.

Such a triangle is ABC (fig. 50); CB is parallel to W, and represents it; BA is parallel to P, and represents it; and AC is parallel to R, and represents it. If, then, the position of the ladder were given us, and also its weight, we should simply have to draw the above diagram, and measure the sides of the triangle ABC, in order to determine the pressures P and R in terms of W; the direction of R being also given by measuring either the angle BAC or BCA.

This would be solving the problem by construction.

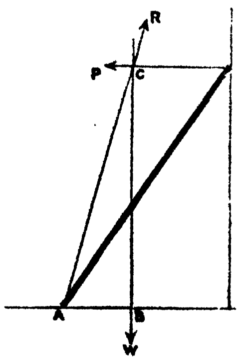


Fig. 50.

140. But suppose we wished to do it by calculation, applying the general conditions of sect. 138: we should first consider the inclined force R resolved into two (see fig. 49), a normal pressure N, and a friction F (the friction being always in such direction as best hinders slipping, sect. 115), and then say that, since there is equilibrium

as regards translation, there can be first no up or down resultant, or N and W must be equal and opposite; and then that there can be no horizontal components, or F and P must be equal and opposite.

But to determine either F or P, in terms of W, we must make use of the second condition—the condition for no rotation—namely, that the forces can have no rotating power, or resultant moment, about any point. Take it numerically:

Let us suppose that we are told the weight of the ladder is 60 lb., and that its centre of gravity is $\frac{1}{3}$ of its length up, that the foot of the ladder stands six feet from the wall, and the top of the ladder thirty feet from the ground; then, as the condition for no translation, we have already found

$$N = W,$$

and

$$F = P.$$

But we don't know either F or P yet; we must find them by taking moments about some point—any point we like, for we know that since there is no rotation the sum of the moments about every point must be zero.

Suppose we take moments about the point A, then neither N nor F has any moment; so the moment of P, $P \times 30$ feet, must be equal and opposite to the moment of W, $W \times \frac{1}{3}$ of 6 feet.

hence $$15P = W = 60 \text{ lb. weight},$$

or $$P = \text{the weight of } 4 \text{ lb.}$$

And we already know that F and P are equal; so then N, F, and P are all known, and now too we know R, because $R^2 = N^2 + F^2$; that is, $R = 60{\cdot}13$ lb. weight.

See if this agrees with a determination by measurement, and then repeat the whole process with the wall rough instead of the ground, and then with both wall and ground rough.

If the inclination of the required R to the vertical in fig. 49 be greater than the limiting angle of friction, equilibrium is impossible, unless a wedge be placed under

the foot of the ladder, which may be considered as equivalent to tilting the ground up. If the ground is like a sheet of ice, the required force F can be supplied by a rope tying the ladder to the wall. If the ground is level and smooth, and if no extra force is applied to the foot of the ladder, equilibrium cannot be attained by any roughness of *wall* short of actual attachment, as by a hook, because N and W will not intersect except at infinity.

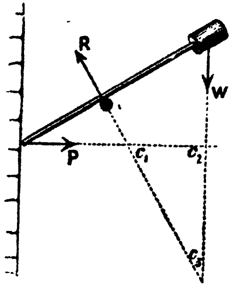


Fig. 51.

141. Next consider a weightless rod resting against a smooth wall over a smooth rail, and with a weight stuck somewhere on it, as shown in fig. 51. (A *section* only of the rail supporting the rod is shown, as a small circle.) To determine where the weight must be for equilibrium, the forces acting are: the weight, W; the normal pressure of the wall, P; and the normal pressure of the rail, R.

Now, here again are three forces, so to be in equilibrium they ought to intersect in a point; but in fig. 51 they *do not* intersect in a point, produce them as much as you like; their direction encloses a triangle $c_1c_2c_3$ instead. Hence there is no equilibrium, and cannot be until the three points c_1, c_2, c_3 coincide in one point C.

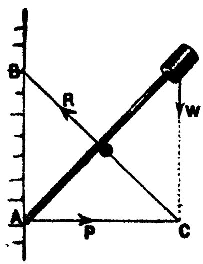


Fig. 52.

Observe that no alteration of *magnitude* in any of the forces can assist equilibrium; a shift of either the direction or the position of some force is essential. The easiest thing to shift is the load; so to find where it ought to be shifted to, draw a fresh figure, and from C, the intersection of P and R, draw a vertical; this will cut the rod

at the point where the weight ought to be placed for equilibrium (fig. 52).

To measure the relation between the magnitudes of the forces when the weight is in this place, we can produce R till it cuts the wall in B; then the triangle ABC has its sides parallel to the three forces,

BA to W,
AC to P,
CB to R;

hence the lengths of these sides will give the forces, if one of them, say W, is known.

It is easy to see that $R^2 = P^2 + W^2$.

In fig. 51 the rod would be slipping up the wall and falling over the rail; this is because the line of W falls to the right of the point c_1, where P and R intersect. If the line of W fell to the left of this point, the rod would slip down the wall, and drop between it and the rail. There is just one position where it does not slip either way; but it is unstably balanced, because motion either way would allow the weight to get lower. Of course if there was any friction, there would be a margin of stability.

142. Now consider a body on an inclined plane held still by some force P acting in any given direction, as depicted in fig. 64, with a diagram, fig. 53. There are three forces, P, R, and W, in equilibrium (R being the normal pressure of the plane), hence P must be in the plane of the other two. To find its magnitude: take off a length AB to represent the weight of the body, and from B draw a line parallel to P, till it cuts R produced in the point C. Then we have a triangle ABC (fig. 53); BC represents P, and AC represents R; and it is easy to measure these lengths on the same scale as AB was drawn.

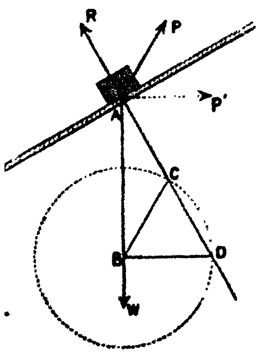


Fig. 53.

There are in general two positions in which the same force P can hold up the body. For draw a circle with centre B and radius BC, it will cut R in two points, C and D; hence the same force P would be just as effective if it acted in a direction shown dotted as P′ parallel to BD; but the pressure on the plane would then be greater than AC, namely, AD.

There is one case when only one direction will do, and that is when the radius of the circle is so small that it only just *touches* R. This radius then represents the minimum force possible, and shows that it must act perpendicularly to R and therefore parallel to the plane, and must have the same relation to the weight that the height of the plane has to its length. If the plane be rough, friction is such a force.

If the applied force P is smaller than this—that is, so small that the line representing it is unable to reach across from B to the line R, then there cannot be equilibrium; and even if P is greater than this, but does not act in the best direction, there need not be equilibrium, and the body will slide down, as in fig. 54: the accelerative or resultant effective force being the component of W along the plane—namely, AM, minus the component of P along the plane—namely, AN. The pressure on the plane is the component of W at right angles to the plane, minus the component of P at right angles to the plane; that is, A*m*—A*n*.

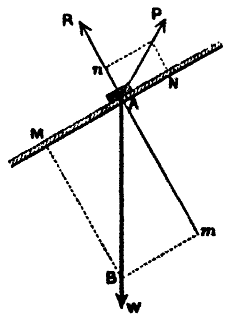


Fig. 54.

When the plane is rough its reaction R need not be normal, but may be inclined at any angle up to the limiting angle of friction on either side of the normal. There will thus be two critical cases—one when the body is on the point of sliding down, the other when it is on the verge of being hauled up. The same construction as above (fig. 53) suffices to determine the needful applied force corresponding to one or other of these

two conditions—namely, produce either limiting R backward and bridge the gap between it and B with a line in the desired direction (fig. 55). If no force at all is needed to hold the body up, the reaction R is vertical; and if the body is on the verge of slipping down, the tilt of the plane is the angle of repose. The least force necessary in any given case is the perpendicular from B on to the line of R; hence the least force is inclined to the plane just as much as R is inclined to the normal, or the best angle of traction is the angle of repose. If the produced line of R ever lies to the left of B, it means that some force is necessary to drag the body down; the inclination of the plane being in that case less than the angle of repose.

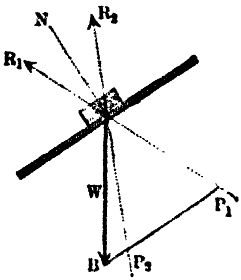


Fig. 55.

143. We have here considered only the slipping of the body; but if it were a ball it would roll, and if it were a block it might topple over, before it began to slide. Let us just see how soon a rectangular block on a rough inclined plane will topple over.

We know that the resultant of all the forces which gravity exerts on the particles of the body passes through the centre of gravity—that is, the body acts statically as if its weight were all concentrated at the centre of gravity. Hence if this point be supported, the whole body is supported. The line of W is the vertical through G; and if this line falls inside the base,* the body cannot topple over; it can only slide

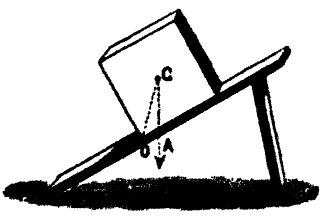


Fig. 56.

* By 'the base' must be understood the area enclosed by a string stretched round that part of the body which touches the plane: consider, for example, the case of a retort stand with a forked foot.

down. To upset the body, it must be tilted through the angle AGO (fig. 56); and if it be momentarily tilted through less than this, it will return to its old position. The angle AGO is a measure of the 'steadiness:' the larger this angle, the more steady is the body. If the vertical through G fell through O, the body would be balanced in unstable equilibrium, and directly the vertical passed outside the point O, the body would topple over; and this applies universally.

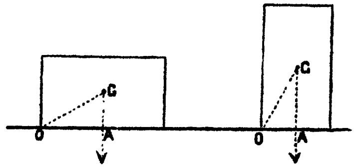


Fig. 57.

A wagon going along with one wheel in the gutter does not upset so long as the vertical through its centre of gravity falls inside the wheel-base; but the act of going over a stone may tilt it sufficiently to make this line pass beyond the base, and then it upsets.

The two bodies in fig. 57 resting on a flat plane are both evidently steady, but the steadiness of the first is much greater than that of the second; and this for two reasons, firstly, because its base is wider, secondly, because its centre of gravity is lower.*

The centre of gravity of an omnibus full outside, but with no inside passengers, must be very high up; and a moderate shock might be sufficient to destroy its stability and upset it.

A block resting on a level surface can be upset by a horizontal force applied high enough. The criterion is obtained by considering moments about the forward edge of its base; if the moment of the force is equal to the moment of the weight about this edge, the body is on

* The most useful measures of steadiness are 1st, *the moment of stability*—namely, the moment of the couple required to upset the body, or the weight of the body multiplied by the distance OA; and 2d, the *dynamic stability*—namely, the work that is required to upset it, or the weight of the body multiplied by the difference of the distances AG and OG (fig. 57).

the point of tilting over. If the force applied is sufficient to call out the maximum friction, the body is on the point of sliding; and at one particular altitude both conditions can be satisfied simultaneously. It is easy to show that this critical altitude for a symmetrical block is half the base divided by the coefficient of friction.

Stability and Instability of Equilibrium.

144. A body in equilibrium, with infinitely small stability, is said to possess *unstable* equilibrium; the least shock must upset it.

Thus, if you narrow the above block till its base is nothing, there remains only a plane or line standing on its edge, and though, when vertical, the centre of gravity of this does not fall without its base, and therefore it is in equilibrium, yet the slightest breath will upset it.

This is not the case with all bodies balanced on a point. A body with a rounded base resting on a plane may be *stable* enough though it cannot be called *steady*. Bodies supported by a point, whether slung like a pendulum or pivoted like a compass-needle or rolling like the half of a split billiard-ball, are said to be in equilibrium; and these examples are in stable equilibrium, because, if you rock them, they will return to their original position after a few oscillations (see sect. 146).

Any segment of a sphere less than the whole sphere will so rock. And a leaden hemisphere would rock in stable equilibrium even if loaded above with a bulky pith figure.

It is quite possible for a body to possess an equilibrium which is neither stable nor unstable—that is, the body, when disturbed, neither topples over nor returns to its original position. All that is necessary is that the vertical

through G shall *always* pass through the point of support, as in the case of a sphere on a flat table; or that the centre of gravity itself shall be supported, as in a flywheel. The body is then indifferent how you place it, and its equilibrium is called *neutral.*

An egg lying on its side has neutral equilibrium for rolling, and stable equilibrium for 'pitching;' it is unstable all ways when balanced on its end.

145. In the case of a body pivoted at a point, if the point is *above* the centre of gravity, the equilibrium is perfectly stable; if *at* the centre of gravity, it is neutral; and if *below*, it is unstable.

Fig. 58.

EXAMPLES.—The nearer the centre of gravity of the beam of a balance is to the point of support, the more sensitive is the balance; but it is necessary to have the centre of gravity slightly lower than the point of support, or the equilibrium would not be stable.

If the balance is to be equally sensitive for all loads, its three knife-edges, the one supporting the beam and the other two supporting the pans, must be in one straight line; the restoring force is then simply the moment of the displaced centre of gravity of the tilted beam. If the weight of the beam W is displaced a minute horizontal distance x, by slightly unequal loads P and Q in the pans, each supported at a distance a from the fulcrum, the equation of equilibrium is $(P-Q)a=Wx$; and of course the pressure on the fulcrum is $P+Q+W$.

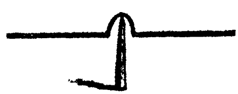


Fig. 59.

A compass-needle is always made with a little central cap, into which the point supporting the needle passes from below, so as to be above the centre of gravity of the needle. See fig. 59.

Again, it is easy to balance a curved beam on a knife-edge, while a straight one will not remain balanced for more than a

few seconds, unless loaded. Compare the diagrams in fig. 60. The weights in the third must be rigidly attached to the beam by rods, not by strings.

Stable. Unstable. Stable.

Fig. 60.

146. In the case of a body with a spherical base standing on a level plane, its centre of gravity cannot help being above the point of contact with the plane, and yet the equilibrium may be stable or neutral; as, for instance, in a sphere the equilibrium is neutral, and in a hemisphere it is stable; or again, it may be unstable, as in an egg balanced on one end.

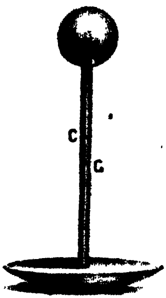


Fig. 61.

The centre of the sphere, of which the base forms a part, is in these cases to be regarded as the real point of support, and then the former rules apply. Thus, if G be the centre of gravity of the combination shown in fig. 61, and if C be the centre of the sphere of which the base forms a part, the whole will oscillate in stable equilibrium.

When a body rolls along any surface, its centre of gravity in general describes a curve with crests and hollows; every hollow corresponds to a position of stable equilibrium (the centre of gravity is then in one of its lowest positions); every crest corresponds to a position of unstable equilibrium, and a measure of the *instability* is the curvature (see sect. 13) of the path of the centre of gravity. For instance, in the case of a body balanced on a point, the higher the centre of gravity above the point the less curved will be its path, and the less unstable will be the equilibrium: for example, it is easy to balance a stick loaded at one end on one's finger if the load be at the top of the stick, but if the stick be inverted it is not easy.

The criterion for equilibrium, as well for its stability

or instability, obtained by considering the path of the centre of gravity, is a useful and very general one. When weights are in equilibrium, as on an inclined plane or on a system of pulleys, it is because the path of their common centre of gravity is horizontal if it move at all. If the path is a horizontal straight line, the equilibrium is neutral and a slight shift makes no difference. It is an instructive exercise to prove that in the case of weights in equilibrium on an inclined plane the path of their centre of gravity is a horizontal line.

If the path of a centre of gravity is curved, the equilibrium is stable or unstable according as it curves upwards or downwards.

An instructive example is afforded by a wooden disc loaded with lead near its circumference. Such a cylinder placed on a slightly inclined plane may easily roll uphill into a position of equilibrium with its centre of gravity vertically over the point of support.

Metacentre.—In cases where a small disturbance changes the point of support, like the case of a portion of a sphere on a level table, there is usually a point in the body through which a vertical through the point of support (what may be called the line of support) will continue to pass, at least if the disturbance be only small. This point is the intersection of the new line of support with the old line of support, if the latter be considered as marked in the body and rotating with it, and it is called the *metacentre*. In the case of a sphere, the centre of the sphere is its metacentre. The conditions of equilibrium for small disturbances are the same as if this point were a pivot, and its height above or below the centre of gravity measures the stability or the instability of the equilibrium. For bodies of irregular shape it does not follow that the successive lines of support intersect at all, and then there is no metacentre. Often there are two, one for a rolling or broad-

side motion, the other for a pitching or lengthways motion. For further information on this subject, see sect. 179.

147. Criterion of equilibrium by zero work done.—A powerful method for determining condition of equili-

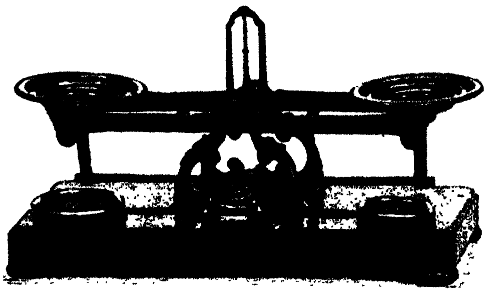


Fig. 62.—Letter-balance.

brium is sometimes called the method of 'virtual velocities,' and consists in observing that if a body subject to any forces is in equilibrium, it must be true that when slightly shifted in any way no work is done; that is to say, if every force is multiplied by the velocity of its point of application when a displacement is made, or by the virtual velocity of this point when displacement is only imagined, the products will all add up to zero.

It is interesting and easy to show that in cases of levers, inclined planes, pulleys, &c., worked by gravity, only the centre of gravity of everything remains unchanged in vertical height whenever there is smooth equilibrium.

Again, in the case of any parallel motion, such as is employed to support the platform of railway weighing-machines and in the ordinary scale letter-balances, the position of the thing to be weighed is unimportant; and it is unimportant because wherever the load is, on any parallel-moving scale-pan, there will be the same vertical descent, and therefore the same virtual velocity, and the same work done.

EXAMPLES—XVIII.

(1) When a weight is supported on an inclined plane by a force acting along the plane, show that the ratio of the force to the weight is the same as the ratio of the height of the plane to its length.

(2) And show that the ratio of the supporting force to the normal pressure on the plane is the same as the ratio of the height of the plane to its base.

(3) Hence show that if a body is supported on a plane only by friction, it will begin to slide down when the ratio of the height of the plane to its base is equal to the coefficient of friction (see sects. 142 and 115).

(4) A picture-frame weighing 10 lb. is hung by a cord passing over a nail, the two parts of the cord making an angle of 120° with each other. Find the tension in the cord.

(5) If the two parts of the cord included an angle of 90°, what would then be the tension?

(6) If a rod rests inside a smooth spherical shell, its centre of gravity must be vertically under the centre of the sphere. Hence, if the rod be uniform, it can only lie horizontally, unless it is equal in length to the diameter of the sphere. Verify these statements.

(7) It is wished to upset a tall column by means of a rope of given length, pulled by men on the ground. At what height above the base of the column will it be best to attach the rope?

(8) A uniform rod hanging from one end is pulled aside from the vertical by a horizontal force equal to half its weight applied at its lower end. At what angle will it be in equilibrium?

(9) A simple pendulum is pulled aside from the vertical by a horizontal force equal to half its weight. At what angle will it be in equilibrium?

(10) A bar of uniform thickness inclined at an angle of 30° with the horizontal, with one end against a wall, rests across a rail at a point 2 feet away from that end. Find the length of the bar if the rail and wall are both smooth.

(11) If the bar is 8 feet long and weighs 10 lb., and the rail is smooth but the wall rough, show by help of a diagram the direction and magnitude of the pressure of the bar against the wall.

(12) Construct a diagram for the case of a bar in a given position, but with an adjustable load, supported by a rough wall and rail; (*a*) when the bar is on the point of falling over the rail; (*b*) when it is on the point of slipping down between rail and wall.

(13) Given distance of rail from wall and length of bar, as *a* and *b* respectively, find its position of equilibrium if perfectly smooth.

(14) A ladder 30 feet long, whose centre of gravity is 12 feet from the lower end, rests on rough ground against a smooth wall, which is 10 feet distant from the foot of the ladder. Draw a figure to scale showing the directions of the reaction of the wall and of the ground; and find by measurement the magnitudes of these reactions if the ladder weighs 80 lb.

(15) Check the above by calculating the result by the method of moments, and determine the coefficient of friction.

(16) Show that a light ladder ascended by a heavy weight is not safe if inclined at an angle greater than the angle of repose.

(17) A beam 10 feet long weighing 80 lb., with its centre of gravity 4 feet from one end, is suspended horizontally by two strings attached to its ends, and fastened to two pegs in a horizontal line 16 feet apart; the shorter one of the strings is 5 feet long. Find the length of the other string, and the tensions in the two strings.

(18) A rectangular block of oak 8 inches long, 3 inches broad, and 3 inches thick, rests with one of its square faces on a horizontal oak surface. If the surface be gradually tilted, how will the block begin to move? [Take the coefficient of friction as 0·4.]

(19) If it stand on a level surface with coefficient of friction $\frac{1}{4}$, and if it weigh 28 lb., what force would make it uncertain whether to topple over or slide along?

(20) A solid consists of a hemisphere with a cylinder standing centrally on the base of the hemisphere; the radii of the hemisphere and cylinder are respectively 6 and 3 inches, and the height of the cylinder is 6 inches. Find the position of the centre of gravity of the solid.

(21) If the above solid is placed on its hemispherical end on a horizontal table, will the equilibrium be stable or unstable?

(22) If the same solid stands on its cylindrical end, through what angle must it be turned to upset it?

(23) A wooden cylinder, 1 foot in diameter, with some symmetrical holes of two inches diameter already bored through it lengthways at a margin of one inch from the circumference, is found to weigh 3 lb., and one of the holes is then filled up with 3 lb. of lead. Find the position of the centre of gravity, the restoring moment when the cylinder is rolled through 90° on a flat surface, and also its position of equilibrium when on an inclined surface.

(24) On what slant would this cylinder be on the verge of rolling down? Construct its position of equilibrium on a plane rising 1 in 6.

CHAPTER IX.

ON MACHINES AND OTHER CONTRIVANCES ILLUSTRATING THE FOREGOING PRINCIPLES.

ELEMENTS OF APPLIED MECHANICS.

148. A machine is an instrument for transferring energy in such a manner that certain useful or desirable work is done. The effective force exerted by the agent which loses the energy used often to be called the 'power,' the force exerted by the body which receives it being called the 'weight;' but the terms 'effort' and 'resistance,' introduced by Rankine, are better. The machine is simply a mediary by which the energy is indirectly transferred from one body to the other.

The quantity of energy gained by the one body is equal to that lost by the other, except for what may become dissipated as heat or other non-mechanical form of energy; in other words, no increase in quantity of energy is ever effected by any machine.

Numerous attempts have been made to construct a machine able to effect this: such attempts are called the search after perpetual motion, and always result in failure (cf. sect. 88). All that one can do by means of any machine is to vary the *ratio* of the two factors, F and s, occurring in the product work, the product itself remaining unalterable. But just as the number 12 may be split up into various pairs of factors, 12 and 1, 6 and 2, 3 and 4, so the factors of the constant product work may be varied at will: and this is the use of a machine. Given a force,

and a distance through which it can act, a machine can always be devised to overcome any other force whatever through some definite distance, such that the product of the second force and distance is nearly equal to the product of the first force and distance. The greater the force required to be overcome, the smaller is the distance through which it can be overcome by a given expenditure of energy. In other words, a feeble agent moving quickly may be able, by means of a machine, to overcome a great resistance, that is, may move slowly an obstacle of considerable strength; and the slowness will be proportional to the force to be overcome. This is often expressed by saying that what is gained by any machine in power is lost in time (or in distance). Or again, by saying that the 'mechanical advantage' of a machine—the ratio of the resistance overcome to the least force required—is the inverse of the ratio of the distance travelled by the 'resistance' to the distance travelled by the 'effort;' supposing the machine to be perfectly efficient, having no friction. A brief way of stating the same thing is to say that the 'force-ratio,' which is another name for 'mechanical advantage,' varies inversely with the 'speed-ratio.'

This condition may also be expressed by saying that if any system in equilibrium under the action of any number of forces receive a slight displacement, then the total work done by the whole of the forces, or the total loss of potential energy, is zero. In other words, the sum of the products of all the forces into the respective distances they have simultaneously moved, or, what is the same thing, into the respective velocities of their points of application measured along their lines of action, is zero. This is frequently a useful mode of finding the condition of equilibrium of a system, and it has already been so applied in sect. 147.

It is often referred to as the principle of *virtual work* or *virtual velocities*; the meaning of the word 'virtual' being merely that the displacement or shift supposed to take place is an imaginary one, and need not really occur.

Efficiency.—If a machine could be arranged so that the

work done by the one body were equal to the energy gained by the other, the machine would be called perfectly efficient —that is, its efficiency would be unity. But in practice there is always some loss of energy by friction, &c., and so the efficiency is a proper fraction; and the ratio of the energy recovered to the energy put in is a measure of the 'efficiency' of the machine. It will be seen from this definition of efficiency that it is equal to the product of the force-ratio and the speed-ratio.

Simple Machines.

149. A pulley is a simple machine by which a weight may apparently be supported by means of a force only half as great as itself; the obvious reason being that the other half of the force necessary to support the weight is supplied by the hook fixed in the ceiling, to which one end of the cord is attached (fig. 63). If the force P exceeds in the slightest degree half the weight, it must raise it; but only half as fast as itself ascends. To raise it at the same rate would require *both* parts of the loop of cord in which W is slung to be lifted. If only one end is lifted, the wheel or pulley rotates, and W only rises at half the rate.

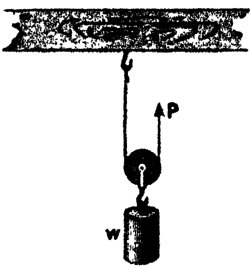


Fig. 63.

The *mechanical advantage* or 'force-ratio' of a simple pulley is thus 2, and its 'speed-ratio' is $\frac{1}{2}$.

An inclined plane is another simple machine on which a weight may be apparently supported by a force less than its own weight; the reason being that the rest of the necessary force is supplied in a stationary manner by the pressure of the plane. If the sustaining force or 'effort,' P, is applied as shown in fig. 64, it is evident that a descent of P through a vertical height l, equal to the whole length of the

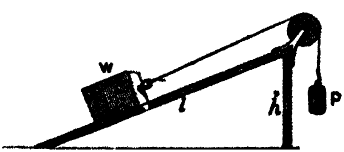


Fig. 64.

plane, would pull W all the way up the plane indeed, but would only *raise* it a vertical height h; hence the mechanical advantage of this machine is $\frac{l}{h}$; and if P exceeds $W\frac{h}{l}$ in the slightest degree, it must raise the weight; provided, of course, that there is no friction.

A lever, a wheel and axle, and a capstan are simple machines in which a weight applied at a great distance from an axis of rotation may apparently support a greater weight nearer the axis; the reason being that the rest, or the whole, of the sustaining force is supplied by the support of the axis, or the *fulcrum*.

Thus, in the left-hand diagram of fig. 65, P and W are both really supported by the fulcrum F; the pressure on it being always $W+P$; in the right-hand diagram the weight is sustained by F and P jointly, and the pressure on the fulcrum is $W-P$. All that P does is to balance the *rotation* tendency of W; and for this purpose its moment, $P \times AF$, must equal the moment of

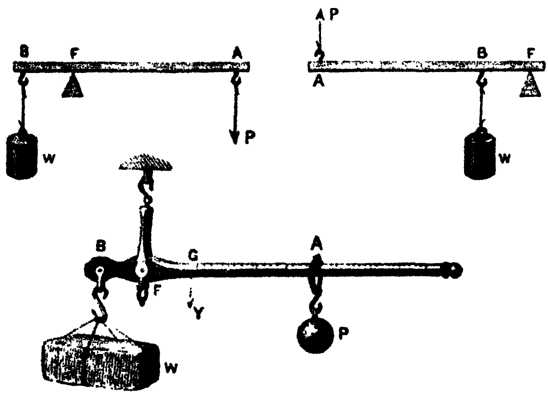


Fig. 65.

W, $W \times BF$. Hence the mechanical advantage of a lever, the ratio of W to P, is always $\frac{AF}{BF}$, or the ratio of the 'arms' of the lever.

In the case of the steelyard (fig. 65), the weight of the 'yard,' Y, acting at its centre of gravity, helps the smaller weight P, so that $W \cdot BF = P \cdot AF + Y \cdot GF$.

A lever cannot, however, be used to raise weights *far;* but an easy modification, securing continuous action, is to make the fulcrum F into a pivot, and to apply P and W at the circumferences of circles or wheels, with common centre F. Thus we get the wheel and axle, or capstan (fig. 66), of which the mechanical advantage is, as before, the ratio of the distance of P from the pivot to the distance of W from the pivot—that is, the radius of the wheel divided by the radius of the axle.

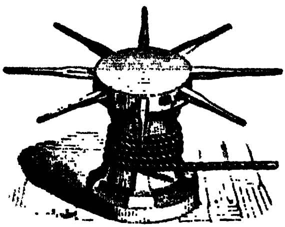


Fig. 66.

Combinations of Simple Machines.

150. Any of these machines may be combined together, so that the resistance of one machine constitutes the 'effort' of the next, and the mechanical advantage of the combination will be the product of their separate mechanical advantages.

Thus three pulleys are shown combined in fig. 67, and the

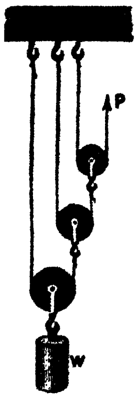


Fig. 67.

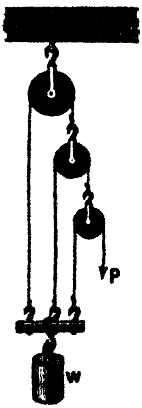


Fig. 68.

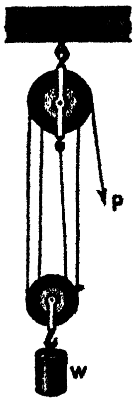


Fig. 69.

mechanical advantage of the combination is $2 \times 2 \times 2$, or 8, if the pulleys are weightless. If W is raised one foot, P must rise eight feet. The whole pull on W is here the pull of the beam

above plus P; hence the pull on the beam is W - P. The arrangement may evidently be turned upside down, so that the beam becomes the weight, and the weight the beam (fig. 68). In this case the weight supported is less by P than it was in the former case. Fig. 67 used often to be referred to as the first system of pulleys, and fig. 68 as the third.

If the weight of the pulleys is not small enough to be neglected, call them w_1, w_2, &c., and consider fig. 67. The lowest pulley is attached to the weight, and rises at the same rate as it does; the second pulley rises at twice, and the third at four times this speed. Now, if any weight w be raised a height h, the work done is wh: so if W is raised one foot,

$$W + w_1 + 2w_2 + 4w_3$$

represents the whole work done by P, in moving through a distance of 8 feet, that is, by the expenditure of 8 P units of potential energy; hence, in general when there is equilibrium, the mechanical advantage W : P must be determined from the equation,

$$2^n P = W + w_1 + 2w_2 + 4w_3 + \ \ldots\ldots \ + 2^{n-1} w_n,$$

if there are n pulleys. This equation expresses the fact that the algebraical total of the work done is nothing; or, if we choose to put it so, that the common centre of gravity remains fixed in position, or at any rate does not rise or fall.

The only one of the old-fashioned systems of pulleys frequently employed for hoisting is what was called the second system, where there are two blocks of pulleys, one attached to the weight, and the other to the beam; and where the same rope passes round all (fig. 69). The mechanical advantage in this case is simply equal to the number of strings supporting the weight: which in the figure happens to be four.

151. A combination of levers is sometimes used, but more often for the purpose of magnifying small motions than for exerting great force; that is, for increasing the factor s in the product work at the expense of the factor F. In fig. 70 the motion of the screw is magnified, the pointer describing a considerable arc for one turn of the screw; such an arrangement is sometimes employed for measuring expansion of a rod by heat, as in the so-called Ferguson's pyrometer: the screw-support being there moved far back, and the rod inserted horizontally between the screw and the lever. A screw alone may be regarded as a

combination of a lever and an inclined plane, the inclined plane being coiled up into a spiral or screw-thread (fig. 71). For every complete revolution of the lever, the resistance is overcome through a distance equal to that between the spires of the screw-threads; hence the mechanical advantage of a

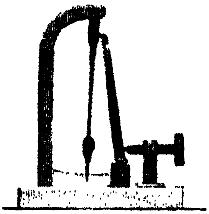


Fig. 70.

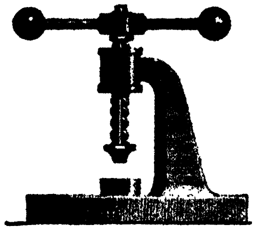


Fig. 71.

screw-press is the circumference of the circle traversed by the force applied at right angles to the lever, divided by the 'pitch' of the screw—that is, the distance between its successive spires. Wheels and axles are usually combined by means of cogs, as is well seen in the wheel-work of a clock.

A pulley is often used in conjunction with a capstan, the rope passing round a pulley attached to the weight, and the mechanical advantage of the capstan being thereby doubled. Moreover, the free end of the rope, instead of being rigidly fixed, may be coiled round another smaller axle with the same centre F, so that its tension shall help the force P (fig. 72). By this means the mechanical advantage can be increased to any desired extent, for the weight is now wound up only because the cord wraps itself on to one, the larger, axle faster than it unwraps itself from the other smaller axle; and the two axles may be as nearly the same size as one pleases. The mechanical advantage is the radius of the wheel (or the length of P's arm) divided by the difference of the radii of the two axles, the whole being multiplied by two because of the pulley.

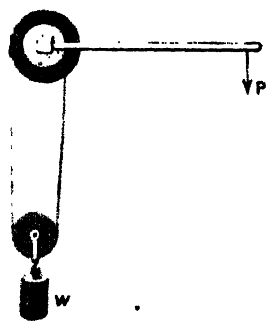


Fig. 72.—Chinese Capstan.

One of the most useful forms of pulley block, the 'Weston' or differential pulley, depends on this principle. It is shown in diagram and in actuality in fig. 73. An endless chain is employed, and the wheels have teeth or ridges gripping links to prevent slipping. The two wheels are of unequal size, and when the chain is pulled, one of its loops increases in size and the other decreases, so the weight can be hauled up. Its mechanical advantage is twice the radius of the big wheel divided by the difference of radii ; or, what is the same thing, twice the number of teeth in the big wheel divided by the number of its teeth in excess of the smaller one. But as the teeth differ in number usually by one only, the mechanical advantage is usually simply twice the number of teeth in the bigger wheel. It is a convenient feature of all machines with a great mechanical advantage that the friction is able to sustain the load without any applied force.

Fig. 73.

A wheel and axle may be combined with a screw, as shown in the contrivance of fig. 74. When the handle is turned, the screw-thread on its axle sends the cog-wheel forward one tooth for every revolution. Such a screw, which itself does not advance in a nut, but which merely rotates in ordinary bearings, is called an 'endless' screw. If l is the length of the handle arm, n the number of teeth in the wheel, and r the radius of the axle on which the rope winds itself, the mechanical advantage of the whole machine is $\frac{2\pi nl}{2\pi r}$ or $\frac{nl}{r}$.

152. To drive a machine an agent must expend energy upon it, and its rate of expenditure of energy is called its 'power.'* But when the agent is inanimate (like running water or compressed steam), its utilised power is often spoken of as the power of the machine driven by it. The

* Or sometimes its *activity*. The word 'power' is frequently used to express the maximum activity of which an engine is capable ; the actual power at any instant is best called its 'activity,' for an engine of 20 horse-power may be idle sometimes. The use of the term 'power' to denote a force applied to a lever is simple misuse, and can only be tolerated as an old-fashioned usage.

power of a machine, then, means its *rate of doing work*; in other words, it equals the work done in any short time divided by that time—so many foot-poundals per second.

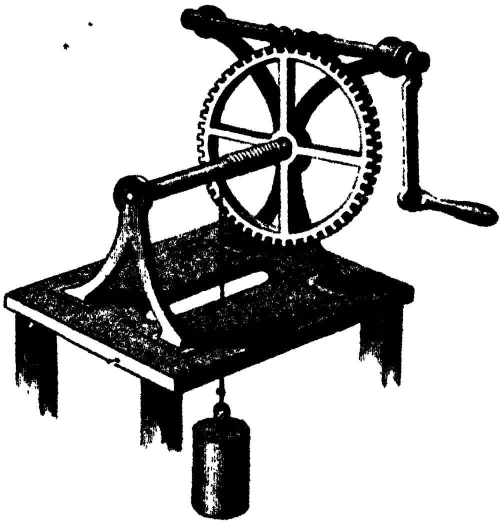


Fig. 74.

A machine is said to have one 'horse-power' when it can do 17,600 units of work every second; which is equivalent to raising 33,000 lb. of matter one foot high against gravity every minute.

EXAMPLES—XIX.

(1) Apply the principle of 'virtual velocities' to determine the condition of equilibrium of a body resting on a rough inclined plane.

The principle is that, if the body receives a slight displacement, the total work done must be zero. The limiting condition required is given in Ex. XVIII. (3), Chap. VIII.

(2) Show that a body on a plane tilted to the 'angle of repose' (see sect. 115) is on the point of sliding.

(3) If a hundredweight be hung on to the hook W in fig. 67,

what force P is required to support it, the pulleys being weightless?

(4) If each pulley weighed 4 lb., what force would be necessary?

(5) In fig. 68 show that if the pulleys are weightless the mechanical advantage is 7; but that if they each weigh $\frac{1}{15}$th as much as the weight, then the mechanical advantage is $8\frac{1}{4}$.

(6) If, in fig. 69, W=20 lb. and P=6, find the velocity of W when it has risen one foot, neglecting friction.

(7) Find also the accelerations of W and of P, and the time required for P to descend 16 feet.

(8) Find the correct position of the weight W in fig. 68, so that the rod on which it hangs may be horizontal. (The figure is not quite correct: the load has to be nearer the string with the greatest tension.)

(9) If four men, each pushing with a force equal to 1 cwt., act at the ends of capstan bars 5 feet long, and wind a rope about an axle 8 inches in diameter, what weight anchor can they raise?

(10) Find the tension in a light flexible rope which is passed round a single movable pulley supporting altogether 20 lb., the free end of the rope passing over a fixed pulley and bearing a load of 12 lb.; and find the acceleration upwards of the 20-lb. weight.

(11) What is the greatest weight a man of 12 stone can raise by means of a Weston pulley block (fig. 73) if the wheels have 12 and 13 teeth respectively? If the friction is equivalent to $\frac{1}{10}$th of the load, what weight hanging on the chain will raise a hundredweight load suspended on the movable pulley of the above system, and what weight hanging on another part of the chain will lower it?

(12) In the first system of pulleys, with 3 equal movable pulleys, a small weight of 7 lb. is found able to balance a large one of 49 lb. Find the weight of each pulley and the tension in each string.

(13) In the first system of pulleys, with three equal movable pulleys, an effort of 20 lb. just supports 146 lb. Find the weight of each pulley, the tension of each string, and the pull on the ceiling.

(14) In the first system of pulleys, with four equal movable pulleys, a force equal to the weight of 20 lb. suffices to

support 245 lb. Calculate the weight of each pulley and the tension in each string.

(15) If a weight of 5 lb. drags up a weight of 8 lb. by means of a single movable pulley, calculate the tension in the cord.

(16) A weight is to be lifted by means of a system of pulleys of the second order: a block of two pulleys is fixed to the weight, and a rope is carried from an upper fixed block of two pulleys round one of the lower pulleys, up round one of the fixed pulleys, then through the second lower pulley and the second upper pulley, and, finally, to the horizontal axle of a windlass fixed to the ground. The diameter of the axle of the windlass is 3 inches, and the length of the handle 18 inches. Find the ratio of the weight (including the lower block) to the effort necessary to lift the weight; and find also the number of turns of the windlass requisite to raise the weight 22 feet.

CHAPTER X.

ON PROPERTIES AND STATES OF MATTER.

(Rudiments of Elasticity, and Introduction to Fluid Mechanics.)

153. The particular kind of effect which a given force will produce in a given piece of matter when it does work on it, depends not on the nature of the *force*, for forces can only differ in amount and not in kind, but on the nature of the *matter*. Matter exists in various states, and has very different properties in each state; and though the principal effects of work, or forms of energy, may be summed up, as stated in the Introduction (sect. 5), under the heads Motion and Strain, yet the kind of motion and the kind of strain produced in different sorts of matter may be very different; and we must now proceed to consider briefly some of the peculiar properties possessed by matter in its different states; inertia and apparently gravitative attraction being properties common to all.

154. Hitherto we have only considered matter in a rigid form insusceptible of strain, but it is time now to summarise the most fundamental facts connected with the production of strains in non-rigid matter by the action of forces.

Strain means either change of size or change of shape.

Change-of-size strain is called **Compression** or **Dilatation**, and the active resistance of matter to it is called **Elasticity of Volume**, or **Incompressibility**.

Change-of-shape strain is called **Distortion**, and the active resistance to it is called **Elasticity of Figure**, or **Rigidity**.

The adjective 'rigid' is applied to all bodies which

strongly resist *any* kind of strain; but the term 'rigidity' is used to denote the measure of the resistance to change of *shape*, while the term 'incompressibility' represents the measure of the resistance to change of *size*.

155. Bodies with high rigidity are called **Solids.** The incompressibility of solids such as india-rubber is much greater than their rigidity, and the same is true in a moderate degree of most solids; but with a substance like cork the reverse is true. By the term 'rigid body' in previous chapters, we have always meant a *perfectly* rigid solid. Such a solid it would be impossible to strain by any finite forces; all its particles would maintain their relative positions unchanged, unless the body were *broken*—for this would be possible; perfectly *rigid* does not mean perfectly *strong*.

Such a solid does not exist, though it is approximated to by rocks and metals. All actual solids are capable of being strained—that is, they all yield somewhat to the action of external forces applied to them; and they are divided into two extreme classes, according to the *way* in which they yield.

They may yield *actively*; the stress exerted by their particles in opposition to the distorting force continuing constant, no matter how long that force is applied, and restoring the body to its old shape the instant the distorting force is removed, without the least *permanent* strain or *set*; in which case they are called *perfectly elastic*. Glass and steel are practically so.

Or they may yield *passively*; passing into any shape without exerting any *continuous* stress in opposition to the distorting forces, and therefore not recovering their form at all when these forces are removed. In this case they are called **perfectly plastic** or **inelastic**; putty, wet clay, and dough are practically so.

Most solids (strictly speaking, *all* existing ones) lie between

these two extremes; they have a certain amount of elasticity combined with a certain amount of plasticity, partly yielding permanently and partly springing back; as you see at once if you bend iron, wood, paper, &c.

156. A great number of things are elastic when the distorting forces are small, but experience a 'set' when they are too great. These are said to be elastic between certain limits, called the **limits of elasticity.** If strained above those limits, they are more or less plastic, and if still more strained, they are torn asunder or broken. The greatest longitudinal stress (sect. 160) which a material can bear is called its **tenacity.**

157. When a solid is strained, both its elasticity of volume and its elasticity of figure are generally called out, for both size and shape usually change. For instance, if you stretch a piece of india-rubber, it alters greatly in shape, but it also expands a little. The strains practically produced in solids may be conveniently considered under the heads of—(1) *longitudinal elongation* or compression; and (2) *shear.*

The first is produced when a rod is either stretched or squeezed lengthways by a simple traction, and the elasticity involved is called *longitudinal rigidity*, or frequently *Young's modulus of elasticity.*

Shear is produced by couples, as when you twist a rod or cut anything with a pair of scissors. It involves the sliding over one another of parallel planes in the body—thus a book is sheared when its top cover is either pressed sideways or turned round, while its lower cover is held still. The sliding of the parallel planes (or leaves of the book) is then well seen, especially if you use a thick book like a London Directory. There is in a pure shear no change of size, only of shape. The elasticity involved in a shear is called *torsional rigidity*, or simply *rigidity.*

When a beam is bent, say by a weight resting on its middle, its lower or convex surface is elongated, and its upper concave surface is compressed, hence longitudinal rigidity only is called out; unless indeed its horizontal planes slide over one another to some extent, in which case simple rigidity will also be brought into play. If you bend a book, you will see that the leaves slide.

158. All resistances to strain are included under the general name Elasticity (the term *elastic* having a slightly different meaning from *elasticity*, just as *rigid* has from *rigidity*).

A body which exerts a great stress when subject to a given strain, is said to have a high elasticity, but if a small stress, a low elasticity; in fact, elasticity is defined as the ratio of the stress called out to the strain which calls it out; or shortly,

$$\text{elasticity} = \frac{\text{stress}}{\text{strain}};$$

or what is the same thing, *the elasticity of a body is measured by the stress called out in it by unit strain.* 'Stress' is here short for *pressure or tension per unit area.* The fact that the above ratio is constant is called Hooke's Law.

159. The *kind* of elasticity depends on the nature of the strain; if it is simple dilatation or compression, the ratio of the stress to the strain is *elasticity of volume;* if it is a linear elongation or contraction, the ratio is *Young's modulus;* if it is a twist or shear, the ratio is *simple rigidity;* and the most general kind of strain that can possibly be given to a body can be compounded of these three elements, or can be resolved into them.

Moreover, a shear may be analysed into two longitudinal strains, a stretch and a squeeze, at right angles to one another; similarly a shearing stress may be resolved into a pressure and an equal tension perpendicular to it.

160. **Strain** is always measured as a ratio; the ratio of a change to an original. The first sort of strain, simple change of size, is best illustrated by gases. See Chapter XIII., Part ii. This strain is measured as the ratio $\frac{\text{change of volume}}{\text{original volume}}$. The second kind, or longitudinal strain, is measured by the ratio of the change of length of a rod to the original length. The third kind, or shearing strain, is

measured by the angular contortion of the body in a direction lying at 45° to the shearing stress.

Stress is measured by the pressure or tension *per unit area*—for instance, the force applied to either end of a rod divided by the area of the cross section of the rod.

161. Notice that elasticity is measured not by the ratio of *distorting force* to strain, but by the ratio of *internal stress* to strain; for a body may be quite inelastic, and yet require a considerable force to distort it. You would find it hard work to flatten out or to punch a hole through a mass of wet clay, for instance; but no active internal stress would be exerted capable of restoring the body to its old shape when the distorting force was removed—the resistance would have been produced by friction between the different parts which have slid over one another, and friction, we know, is a passive force which can destroy motion, but can not generate it.

Such bodies as these then, though *plastic*, are *viscous*—that is, there is friction between their particles, so that energy is converted into heat when they are distorted. Elastic bodies also may be viscous—that is, there may be some friction between their particles whenever *shear* or sliding of parts occurs. Even steel is very slightly viscous, and when bent becomes infinitesimally warmer, otherwise a tuning-fork *in vacuo* could go on vibrating for ever.

162. Matter, however, is known to exist in a perfectly plastic state, which is not viscous at all, but *limpid*; and in this state it is termed *fluid*. A *perfect fluid* is a body with zero rigidity and zero viscosity—in other words, it has infinite plasticity and infinite limpidity. No force whatever is required to alter its shape, but it takes the shape of whatever vessel contains it. Many *actual* fluids come very near to this, but they all have more or less trace of viscosity. Ether has a little less than water, while oil has more, treacle has more than oil, Canada balsam still more, and pitch or

sealing-wax a great deal—so much that it is practically a solid except for very long-continued forces. The only elasticity possessed by fluids is *elasticity of volume;* in other words, no permanent stress is called out in them by any strain except simple expansion or contraction.

163. All fluids are perfectly elastic as regards *volume*—that is, they all regain their size perfectly when the compressing stress is removed. Nevertheless the values of their elasticities vary very much, for some are nearly incompressible, while others are readily compressed; and they are divided into two great groups on this ground.

The group of fluids which have a very high volume-elasticity, or are nearly incompressible, are termed **liquids** —type, water. A *perfect liquid* might be defined as an **incompressible perfect fluid.**

The other or compressible group have an elasticity not depending on themselves at all, but simply on the pressure to which at the time they are subject—the elasticity being proportional to the pressure; these are termed *gases*—type, air. From this it follows that the volume occupied by a gas also depends, not upon itself, but upon the pressure to which it is subject. Gases in fact take not only the *shape*, as all fluids do, but also the *size* of their containing vessel, no matter how large this may be.

We may sum up shortly thus:

Solids have both size and shape.
Liquids have size, but not shape.
Gases have neither size nor shape.

Matter exists in all kinds of states, some approximating closely to one of these three types, others lying between them and passing almost insensibly from one type to another.

164. The only forms of matter which can be treated in a simple manner, besides perfectly rigid and perfectly elastic

solids, are *perfect liquids and perfect gases;* and also ordinary liquids and gases when at *rest.* It remains now to see what special mechanics is necessary for matter in these two fluid states.

The special mechanics for liquids is called **Hydrodynamics**; the branch of it treating of liquids at rest being **Hydrostatics.**

The branch of hydrodynamics relating to liquids in motion, or **Hydrokinetics,** is not an easy subject, though it is a profoundly instructive one. The practical application of liquids in machinery is called **Hydraulics.**

The special mechanics for gases is called **Pneumatics,** or sometimes Aërodynamics.

165. The essential difference between the mechanics of solids and the mechanics of fluids is based upon the different ways in which they transmit pressure. Thus, take a rigid stick standing on the ground, and press downwards upon the upper end of it; the pressure is transmitted unchanged to the other end, which therefore presses the ground with an equal force; but not the slightest pressure is exerted sideways, say against a tube surrounding and fitting the stick. But place some liquid in a closed tube, and press one end of the liquid with a piston; then, though the pressure is still transmitted to the other end, it is also transmitted sideways to every part of the tube just as much; and, moreover, the force experienced by the closed end of the tube is not now necessarily equal to the force applied to the piston, unless the area of the closed end equals the area of the piston; if the area is greater, the force transmitted is greater, and if less, less. Every portion of the surface of the tube which exposes to the liquid a surface equal to the area of the piston, experiences a pressure equal to that exerted by the piston; a fact which is briefly expressed thus:

Fluids transmit pressure equally in all directions.

This is entirely because of their plasticity, or the perfect mobility of their particles. The structure of a liquid might be imitated roughly by a number of exceedingly small well-oiled shot. A bag full of such shot, if compressed in any way, would experience the pressure in every part of it.

EXAMPLES—XX.

(1) A wire of sectional area 1 square millimetre and 1 metre long is stretched 3 millimetres by a load of 6 kilogrammes. What is the Young's modulus of its material?

(2) What stress would lengthen the above wire 1 millimetre?

(3) What stress would lengthen a wire 1 per cent., if its Young's modulus is E?

(4) What stress would shorten a bar three parts in twelve thousand, if its Young's modulus is E?

(5) What load on a vertical iron rod 1 inch square would shorten it by one-thousandth of its length if the Young's modulus of iron is 1 million atmospheres?

(6) If a linear thrust of 60 tons per square foot diminishes the length of a bar by a tenth per cent., and its volume by a twentieth per cent., how much must the rod have temporarily increased in diameter during the thrust?

(7) With the above data, how much diminution of volume would you expect if the bar were subjected to a uniform hydraulic pressure of 60 tons to the square foot all over its surface?

(8) What then is the cubic compressibility of the bar?

(9) What is the incompressibility or volume elasticity of the material of the above bar; also what is its Young's modulus?

(10) The volume elasticity of sea-water is about 20,000 atmospheres. How much compressed is it at a depth of 150 times 34 feet, or say a mile?

(11) How much would an ocean, two miles deep, rise in level if its water became incompressible and resumed its surface density?

(12) If atmospheric air is squeezed one per cent. by the hundredth of an atmosphere applied for some time, what is its slow elasticity?

(13) If the same air is squeezed quickly by the same pressure, it only shrinks at the first instant five-sevenths of one per cent.: what is its quick elasticity?

CHAPTER XI.

ON THE PRESSURE OF GRAVITATING LIQUIDS AT REST.

(*Hydrostatics.*)

166. We conceive a perfect liquid as an incompressible fluid, that is, a body all whose particles are capable of free motion among themselves without the slightest friction, whose shape therefore is wholly indefinite, but whose volume it is impossible to change. Water is an imperfect liquid, partly because it is slightly compressible, but principally because it is slightly viscous—that is, because its particles experience when they slide over one another a certain amount of resistance analogous to friction, called viscosity.

Hence it is that a basin full of water which has been stirred round and round and left to itself, will after a time come to rest. The energy of motion will be wasted by 'friction' against the wet sides of the vessel—that is, it will be expended in warming the water. But because the friction is very small, a particle of water can travel against it a long way before its energy is expended—that is, before the work done, Fs, is equal to the energy to be got rid of, $\frac{1}{2}mv^2$.

167. The friction due to viscosity differs from ordinary friction in that it depends very greatly on the speed of the relative motions; it seems, in fact, to be about proportional to the square of the velocity, and as the velocity vanishes, so does the viscosity-friction. The properties of water, or any other actual liquid *in motion*, are therefore very dif-

ferent from those of the ideal perfect liquid; but when water is *at rest*, there is no friction among its particles, their reactions are all normal, and its behaviour is then identical with that of the perfect liquid. Hence it is that the mechanics of liquids *at rest* (even such liquids as treacle) is so simple; the simple laws of the perfect liquid are applicable to them, for their viscosity does not come into action.

Pressure of Fluids in General at Rest.

168. The general law of pressure common to all fluids, and following at once from the mobility of their particles, is that they act like perfectly smooth bodies (cf. sect. 115); or,

The pressure of a fluid at rest is always perpendicular to every surface on which it acts.

For if the reaction of the surface had any component *along* it, it would be able to move the fluid, which would therefore be *not* at rest.

A second general law may also be stated thus: If a pressure is applied to any area of the surface of a fluid in a full closed chamber, that same pressure is transmitted to every portion of the walls of the chamber of equal area (sect. 165).

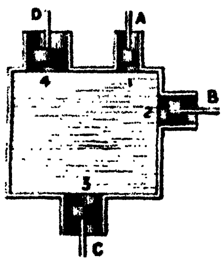


Fig. 75.

Thus imagine a closed cistern quite full of water, with tubes or cylinders let into the sides anywhere, and plungers or pistons, A, B, C, D, fitting these tubes quite freely, but yet water-tight (fig. 75); and let A have an area of 1 square inch; B, 2 square inches; C, 3; and D, 4 square inches. Now push A in with a force say of 20 pounds-weight; every square inch of the interior surface of the cistern will experience this pressure, and therefore B will experience a force of 40, C of 60, and D of 80 pounds-weight. Of

the three larger pistons, let D be the only one free to move, and let a constant external load of 80 pounds be applied to it; then if A is pressed in with a load the least exceeding 20, D will move out and raise the 80 pounds. But it would only move ¼th as fast as A. This is evident; for suppose A were pushed in 1 foot, it would throw 12 cubic inches of water into the cistern, and this water would go into the cylinder of the piston D, if that were the only part of the walls free to move; but as this cylinder is 4 square inches in area, the 12 cubic inches of water would only cause D to move out 3 inches, the quarter of a foot. In other words, the work (Fs) done by the piston A, 20×12, is equal to the work done upon the piston D, 80×3.

So that we have here simply a machine subject to the universal law of machines, that 'what is gained in force is lost in speed;' and there is no gain of energy in a *hydraulic* machine any more than in any other.

The machine just described, put into a working form, is known as the hydraulic or Bramah press (fig. 76). It consists fundamentally of two cylinders of different sizes, with pistons or plungers fitting them, and a pipe connecting them. Water fills

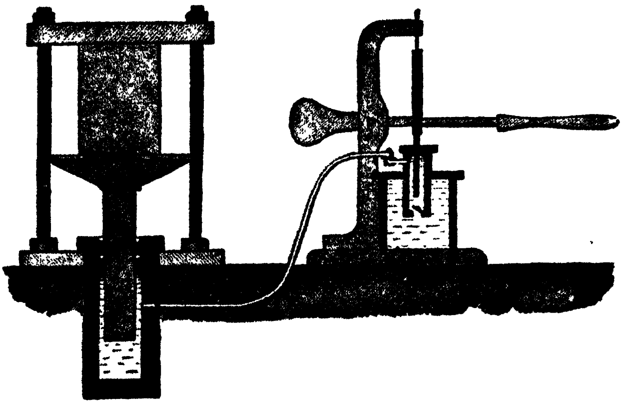


Fig. 76.—Hydraulic Press.

both cylinders, and the mechanical advantage of the machine is the ratio of the areas of the two pistons $A : a$, so that a 50-lb. pressure on the small piston balances $50\frac{A}{a}$lb. on the large one.

The liquid acts only as an incompressible plastic medium for transmitting pressure. For a fuller account of the machine, see Ganot or Deschanel.

169. So far we have supposed the pressure to be produced only by pistons which endeavour to compress the liquid, but it is important to consider also pressures due to the *weight* of the liquid. Every particle of a liquid is attracted to the centre of the earth, and will tend to get there by percolation unless prevented by being enclosed in some vessel with impervious sides; in other words, water must be kept in non-porous vessels. The vessel, however, need not have a lid, for a *liquid* occupies an unchangeable volume, and therefore may have its upper surface free; it keeps at the bottom of the vessel as the nearest accessible position to the centre of the earth. But it will press on the bottom and sides of the vessel with a certain force which will always be normal to those surfaces, and whose magnitude we have now to consider.

Pressure of Liquids due to their Weight.

The first simple law is that the upper or free surface of a liquid at rest is horizontal; that is, is normal to the vertical force of gravity on each particle. Such a surface is said to be *level*, and it is practically flat or plane, because the forces on the several particles are practically parallel.

Inasmuch, however, as these forces are not really parallel, but intersect at the centre of the earth, the level surface of a liquid at rest is not really plane, but is curved round the centre of the earth; in other words, it forms part of a sphere with the radius of the earth as its radius. The curvature is too small to be appreciable in a bucketful of water, but it is apparent enough in the ocean.

Another law, that the pressure of a liquid varies directly with the depth, is what we must now establish.

170. Consider a cylindrical bucket with a flat bottom,

filled with water; the base of the vessel has to support the whole of the water, as if it were a rigid mass slipped into the bucket with its sides well oiled. For although certainly the sides are pressed, and therefore exert reactionary pressure on the water, yet they, being upright, press it horizontally only, and so can have nothing to do with sustaining its weight. The pressures of the sides simply maintain the shape of the water in opposition to the force of gravity, which tends to flatten it out.

The pressure on one side is equal and opposite to the pressure on the other, and therefore there is equilibrium, unless part of one side be removed by boring a hole through it. In that case the water will flow out, and the uncompensated pressure on the side opposite the hole will force the vessel bodily along in a direction opposed to the stream of water. This is the principle of Barker's mill, turbines, Catherine wheels, rockets, &c. See Deschanel, page 92; or Ganot, sect. 193.

In an upright cylindrical vessel, then—that is, any vessel with vertical sides—the pressure on the base is equal to the whole weight of water contained in the vessel. But the cubic contents of a cylinder are obtained by multiplying its height by the area of its base always, whether that base be round or square, or any other shape; and the weight of water a vessel can contain is, of course, its contents in cubic feet multiplied by the weight of each cubic foot. Hence, the pressure on the base of an upright-sided vessel A square feet in area, filled to a height of h feet with a liquid of which a cubic foot weighs s lb., is in lb. weight, $P = sAh$.

Thus, suppose an oblong-based plane-sided cylinder (also called a *prism*) with base 10 inches by 5 inches, and height 15 inches; the contents would be $10 \times 5 \times 15 = 750$ cubic inches, and the pressure on its base when full of water would be the weight of 750 cubic inches of water; which happens to be about $27\frac{1}{4}$ lb. weight.

171. If we are speaking about water, this s is often written w, meaning the weight of a cubic unit of water, just as it might be written m if we were speaking of mercury. Whether w stand

for the weight of a cubic inch, a cubic foot, or a cubic centimetre, is wholly immaterial, being only a matter of custom or convenience; only we must keep to one unit all through. Hence, we use the word, a *cubic unit*, as expressing the cube of whatever arbitrary length happens to be taken as the unit of length in other parts of the book, or question, or problem under consideration.

A cubic foot is found to contain 62·33 lb. avoirdupois of water, which is not far off 1000 ounces.

A cubic inch contains the $\frac{1}{1728}$th part of this—namely, 252½ grains.

A cubic centimetre contains one *gramme* of water; and this is one reason why the French system of weights founded on the gramme makes calculations simpler: the unit of mass, or unit quantity of matter, is defined as that of unit volume of water.

Mercury is 13·6 times as heavy as water. Hence 1 cubic inch contains about $\frac{13600}{1728}$ ounces of mercury; and a cubic centimetre 13·6 grammes.

172. Suppose now that, instead of a cylindrical vessel, we consider a conical one, set up like a tumbler, with the wider end uppermost: then the pressure on the sides, being still perpendicular to them, is no longer horizontal, but has more or less of a vertical component as well as a horizontal one; hence, we can no longer say that the pressure on the base is the whole weight of water in the vessel, for the sides may and do support some.

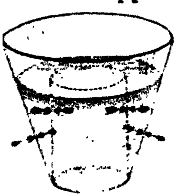


Fig. 77.

How much the sides support, and how much the base, may be readily seen by imagining an infinitely thin circular drum of the same diameter as the base of the vessel to be let into the water, as shown by the dotted lines (fig. 77). Or you may suppose a thin circular drum of the liquid to freeze or become rigid, as indicated by the dotted lines.

The pressure across the walls of this imaginary drum is horizontal; and inside the drum we have what is equivalent to a rigid cylinder, with well-oiled sides, resting on and entirely supported by the base (just as we had in the cylindrical vessel); while

outside we have a ring-shaped mass of water which is not supported by the base at all, and therefore must be supported by the sides. It is, in fact, supported by the vertical component of the pressure of the sides, and therefore it has nothing to do with the pressure on the base, which is wAh as before.

So also, if we turned the conical vessel the other way up, with the wide end as base; the pressure on the base would then be greater than the whole weight of water in the vessel, because of the vertical component of the pressure of the sides, which now acts downwards. And, as the pressure of the water on the sides would, if the sides were removed, be able to sustain the ring-shaped mass of water completing a drum set up on the base, it follows that the whole pressure on the base is still the weight of a volume of liquid filling a cylinder whose base is the actual base, and whose height is the height to which the vessel is filled; or again, wAh as before.

Notice particularly that none of this reasoning is impaired or affected if the sides of the vessel, instead of being plane, are curved or zigzag, or indeed any shape whatever, as in figs. 78 and 79. The pressure on the base is always simply sAh, or the weight of a cylinder of the given liquid with

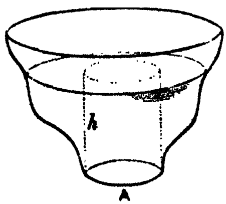


Fig. 78.

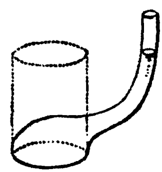


Fig. 79.

the given base as base, and the given height as height; for the base supports this cylinder, the sides support the rest.

173. The vessel shown in fig. 79 is supposed to be flexible like an india-rubber tube, and its base can be turned into different positions as in fig. 80; but, since liquids transmit pressure equally in all directions, the pressure on it will not vary except in so far as the bending of

the tube alters the height of the liquid in it. The only difficulty is the knowing what point to measure the depth to. The pressure on the lower part of the base is greater than

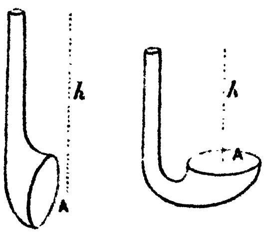


Fig. 80.

that on the upper portion; but since the pressure is simply proportional to the depth, the average or mean pressure will be simply the pressure at the average or mean depth (compare average velocity in sect. 23)—that is, the pressure at the middle point of the base. Hence, the pressure on any surface of area A, immersed under a liquid to the *mean depth h*, is always sAh.

The surface plainly need not form the base of a vessel, but may be immersed anyhow.

Thus, let a rectangular plate 5 inches long by 4 inches broad be immersed slantingly under water, so that its upper edge is 8, and its lower edge 10 inches below the surface. Then evidently its mean depth, or depth of its middle point, is 9 inches; and the pressure on its surface, being equal to wAh, $w \times 5 \times 4 \times 9$ $= 180w = 180 \times \frac{1000}{1728}$ ounces weight.

If the liquid had been mercury, this pressure would have been 13·6 times as great.

To find the mean depth of a bent or curved plate of irregular shape requires calculation, and the calculation required is just the same as that which would be used to find the centre of gravity of the plate (indeed, the centre of gravity is the most middle point in a body); hence the *mean depth* of a surface is often spoken of as *the depth of its centre of gravity.*

So we get the perfectly general result for liquids subject only to gravity:

The total pressure on any plane surface whatever, due to the weight of a liquid under which it is immersed, is its area, multiplied by the vertical depth of its centre of gravity

below the free surface of the liquid, multiplied by the weight of a cubic unit of the liquid;

or in symbols, $P = sAh$.

There is nothing more to explain. This simple formula contains it all.

174. Since the pressure of a liquid does not depend upon the quantity of the liquid, but only upon its depth, we may make a small quantity of liquid exert any pressure we please by putting it in a long narrow vertical tube, and giving it a large area to press upon.

This is the principle of the 'Hydrostatic Bellows;' which consists of a pair of circular boards joined water-tight by corrugated leather like ordinary blow-bellows, with a long tube opening into the cavity between the boards, the tube rising a good height, and finishing off with a funnel. See fig. 81. A man may stand on the upper board of the bellows, and raise his own weight slowly by simply pouring water down the tube.

Fig. 81.

For if A be the area of the upper board of the bellows, and h its vertical depth below the surface of the water in the tube, all that is necessary to balance the man is that wAh shall be equal to or greater than his weight, say 200 lb. or 3200 ounces.

Suppose A is a square foot, then to find the necessary height h to which the tube must be filled, we have $1000 \times 1 \times h = 3200$; or $h = 3 \cdot 2$ feet, a very moderate height indeed.

The man is, in fact, equal to a cylinder of water standing on A as base, and of height 3·2 feet; for this quantity of water would be balanced by the column of water in the tube (see sect. 176 and fig. 82), and the board and man take its place. The man rises so soon as this imaginary cylinder of water is equal to himself in weight; and it will be equal to him in weight just about the same time as it is equal to him in bulk, for a man is just about able to float in water (see Chapter XII.).

Hence the average cross section of a man is equal to the area of the board of a hydrostatic bellows, on which he would just be supported by a column of liquid

equal to himself in height; for instance, if his height were 6 feet, and his weight 15 stone (210 lb.), his average cross section would be ·56 square foot, or 80·64 square inches, because $1000 \times \cdot 56 \times 6 = 210 \times 16$.

175. The total pressure on a surface under a liquid depends partly on itself—namely, on its area; but the pressure *per square inch* of surface depends not at all on itself, but on external conditions—namely, how deep it is immersed, and what it is immersed in: hence it is convenient to distinguish these, and to call the pressure per unit of surface the *intensity* of the pressure, and to denote it by p, so that $p = \frac{P}{A}$; or of course, $p = sh$.

One often speaks simply of 'the pressure of a liquid' at such and such a depth, without specifying the surface on which the pressure is exerted; for instance, the pressure of the ocean at a depth of one hundred fathoms, and so on. In such cases the *intensity* of pressure is always meant, or the pressure which would be experienced by a surface of unit area if placed at that depth—that is, simply sh.

The pressure of an incompressible fluid (or liquid) therefore varies directly with the depth (for s is constant); being nothing at the surface, and increasing uniformly as you descend.

In so far as actual liquids are slightly compressible, this simple proportion between depth and pressure does not hold

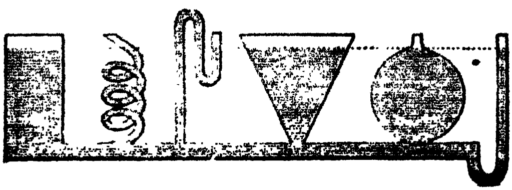


Fig. 82.

down to great depths; the liquid there becomes denser, or heavier bulk for bulk, and accordingly ocean pressure increases rather faster than the depth.

176. When any number of communicating vessels are filled with the same liquid, the level of the liquid in all is the same. See fig. 82.

For the intensity of the pressure at any point due to every column of liquid must be the same, or there could not be equilibrium ; and this pressure is proportional to the depth.

Further, when communicating vessels contain different liquids which do not mix, the heights of the columns of liquid are inversely as their specific weights.

For take any two communicating vessels, say the two legs of a U tube, one full of mercury say, the other of water. Call the area of the surface of contact of the two liquids (fig. 83) A, and let the vertical height of the surface of the water B above A be called h, while the vertical height of C, the surface of the mercury above A, is called h'; then the pressure on each side of the area A must be the same, as soon as there is equilibrium and the columns have ceased to oscillate ; but the pressure on its upper side is wAh, and on its lower, mAh', hence $wh = mh'$, or $h : h' :: m : w$.

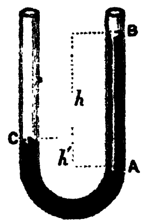


Fig. 83.

This method of balancing columns is employed to find how much denser a liquid is when cold than when hot, the two columns being purposely kept at different temperatures and their heights measured ; and thus the expansibility of a liquid can be measured without having any regard to the size, shape, or expansibility of the vessel containing it.

Centre of Pressure.

177. The whole pressure or resultant force on any flat surface under a liquid may be considered as composed of a number of parallel forces—namely, the pressures on each individual small area of the surface ; and all these parallel forces will have a resultant equal to their sum, passing through a certain point of the surface which is called the *centre* of the parallel forces (cf. sect. 129), or the 'centre of

pressure.' For a rectangular area like a dock-gate this point is $\frac{2}{3}$ of the way down from the surface of the water, because the pressure increases uniformly with the depth. The position of the centre of pressure of an immersed surface corresponds closely with that of the centre of oscillation of a swinging body (sect. 81). If A is the area of a plane surface immersed vertically in a liquid, and O is its centre of oscillation when swinging about the line where the plane, or the plane produced, cuts the surface of the liquid, O is the centre of pressure due to the liquid. In the case of a rectangular plate immersed, with just one edge in the surface, O is two-thirds down the plate.

EXAMPLES—XXI.

(1) The small plunger or pump-piston of a Bramah press is half an inch, and the large one is 8 inches, in diameter; the pump is worked by a handle 5 feet long, the fulcrum being one inch from the point of attachment of the plunger. What is the greatest weight that a man of 15 stone can lift by this machine if he sits on the end of the handle?

(2) Find the pressure on the bottom and sides of a cubical vessel 10 centimetres in the side full of mercury.

(3) Find the pressure on one side of the above cubical vessel if half full of water and half full of mercury.

(4) What is the pressure of water at a depth of 1020 feet?

(5) A couple of hemispheres 1 metre in radius are joined water-tight and placed in water with their join vertical and just submerged. What is the resultant force holding the two halves together?

(6) Find the total pressure on the bottom of a tank 10 feet square, 5 feet deep, and full of water. Find the pressure on a side of the same tank.

(7) A block, in the form of an isosceles wedge, 1 foot high and having a base 1 foot square, is immersed in water with its base horizontal and uppermost at 4 feet below the surface. Determine the pressure, and the vertical component of the pressure on each face. Hence show what must be the weight of the wedge so that it may just float.

(8) A dock-gate 30 feet square has 21 feet and 12 feet depths of

water on its two sides respectively. Find the resultant of the pressures exerted upon it by the water.

(9) A triangular plate is immersed vertically in water with the vertex in the surface and the base horizontal. The height and base of the triangle are each 8 inches. Find the pressure on a face of the plate. Find also the depth of the centre of pressure.

(10) The above plate is immersed vertically with its *base* in the surface. Find the pressure and the centre of pressure on its face.

(11) A cubical tank of 1 foot side is one-third full of mercury and two-thirds full of water. Find the pressure on one of its sides.

(12) What is the pressure against one side of a cubical cistern, when full of water, which will hold 200 gallons?

(13) What is the pressure, in tons-weight per square inch, at the bottom of the sea where the water is two miles deep?

[A cubic foot of sea-water a mile deep weighs about 65 lb.]

(14) In a hydraulic press the pump-plunger is a cylinder 1 centimetre in diameter, and makes a stroke 7 centimetres long. The plunger of the press is 20 centimetres in diameter. Taking the collar friction as a quarter the load in each case, calculate (a) the pressure in the press when a load of 1 cwt. is applied to the pump-plunger; (b) the available force of the press-plunger; (c) the number of strokes which the pump must make in order to raise the press-plunger 10 centimetres.

(15) Find the work done per minute (a) by the operator, (b) by the machine, if the above press is worked at the rate of a stroke a second: the load on the press-plunger being as already calculated.

CHAPTER XII.

FLOATING BODIES (*Hydrostatics continued*).

178. We shall now proceed to consider what happens when a solid is wholly or partially immersed in a liquid. Most of what we shall state will be true of fluids in general, but receives its most obvious illustration in the case of liquids.

When you dip your hand in the water, you displace some of the water; in other words, a portion of space below the surface which was formerly occupied by water is now occupied by your hand. The volume or bulk of the water displaced is, of course, equal to the volume or bulk of your hand.

All solids, then, when immersed either wholly or partially in a liquid, displace a volume of that liquid equal to the bulk of that part of them which is immersed. This is perfectly obvious.

179. Further, when your hand is immersed you can feel, if you attend, a certain pressure urging it up out of the water. This upward pressure is more apparent if you immerse your whole body; indeed the upward pressure is then so great as nearly to counteract the weight of your body altogether, consequently, in a bath you weigh apparently next to nothing.

This upward pressure is what we must now discuss.

Take an ordinary chemical test-tube of very thin glass, and plunge it in water with the closed end downward. You will feel a very distinct upward pressure, and the tube will

be forced up if you let go. Keep, however, the tube immersed, and slowly fill it with water. You will find that it is forced up gradually less and less, until, when the level of the liquid inside and out is the same, the tube will cease to press up, and will weigh pretty nearly the same as it did before it was immersed at all. The displaced water has been restored.

If you perform this experiment accurately with a balance, you will find the tube does not *quite* recover its original weight, even when the level is the same inside and out. It will weigh 60 per cent. of its original weight. This is evidently because some little water is still displaced by the walls of the tube, which, however, are very thin, and in what follows will be assumed to be infinitely thin.

Now imagine the glass tube annihilated; the water it contained will remain occupying the place the tube had occupied, and experiencing the same pressures as the tube did; because the same quantity of water is displaced as before, only now not by the glass tube but by the liquid water which had been poured into it. Obviously, however, this water will be in equilibrium, as all water in water at rest is; hence the two forces under whose influence it is—namely, its weight downwards, and the pressure of the surrounding water upwards—are equal and opposite. But the pressure upwards is the same as the tube experienced before its annihilation; therefore the pressure on the tube was equal to the weight of its own volume of water—that is, the weight of the water it displaced—and acted in the same straight line, namely, through the centre of gravity of the water displaced.

This result is perfectly general, and is known as the principle of Archimedes, because it was the method that he invented when asked to ascertain the chemical composition of an irregular ornamental mass made of two unequally dense metals, without chemical analysis or damage. He

perceived that if its weight and its volume were known its average density could be ascertained, and thus the proportion of its known component metals calculated. He determined its volume by immersing it in water and weighing it there.

When any solid is immersed either wholly or partially in a fluid, it is pressed up with a force equal to the weight of the fluid displaced; and this force may be considered to act at the centre of gravity of the fluid displaced.

The fluid displaced is equal in volume to the solid, hence the upward force is the weight of an equal bulk of the fluid.

To show this by means of our symbols, consider a special case, say a cubical block of stone, a inches in the side, immersed in water, so that its upper surface is at a depth h below the surface of the water, and therefore, of course, its lower surface at a depth $h+a$ (fig. 84). The area of any of its faces is a^2. The pressure on its upper face (sect. 171) is wa^2h, and on its under face is $wa^2(h+a)$, and on each of its sides $wa^2(h+\frac{1}{2}a)$. The pressures on its four sides are horizontal, and are in equilibrium among themselves two and two. The pressures on its upper and lower faces are opposite but are not equal, and therefore are not in equilibrium: their resultant is

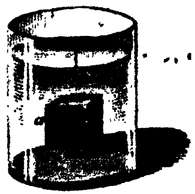


Fig. 84.

$$wa^2(h+a)-wa^2h=wa^3 \text{ units of force}$$

acting upwards. But a^3 is the volume of the block, and wa^3 is the weight of this volume of water—that is, the water displaced by the block; so then the resultant of all the pressures on its entire surface is a single force upwards equal to the weight of the water displaced.

If we did not care about simplicity, the same might be shown by the symbol, for a solid of any irregular shape whatsoever, and a most important mathematical theorem it would be. You may make its acquaintance hereafter in a more general form under the name of 'Green's theorem.'

180. But now we know that if the cube in fig. 84 were really a block of stone it would not stay where it is; it would sink. This is because it is only pressed *up* by the weight of an equal bulk of *water*, whereas it is pulled *down* by the weight of its own bulk of *stone*—which is greater. The resultant force pulling it down, or its apparent weight under water, is

$$sa^3 - wa^3, \text{ or more generally } (s-w)v;$$

if s stand for the specific weight of stone, and v for the

volume of the block, whatever shape it may happen to be. It still weighs downward, therefore, but it has lost weight equal to the weight of its own volume of water. If, on the other hand, it were a block of wood, it would be pulled down only by the weight of the wood, whose specific weight d is less than that of water; consequently it is forced upwards with a resultant force

$$wa^3 - da^3, \text{ or } (w - d)v.$$

And so generally, an immersed body is always urged up or down with a force proportional to the difference of the specific gravities of itself and the liquid in which it is immersed—up, if its specific gravity be the less; down, if it be the greater. Only when the specific gravities of the solid and liquid are equal, does the solid remain floating wholly immersed in any position—that is, in neutral equilibrium.

181. When a light body rises in a liquid, the resultant force urging it up is constant so long as it is wholly immersed; but it decreases as soon as some of the body begins to emerge, and it vanishes as soon as the weight of the water displaced equals the weight of the body. Hence, a body whose specific gravity is less than that of a liquid can float in that liquid, and does float in stable equilibrium when it has displaced a quantity of liquid equal to itself in weight.

A piece of floating wood, for instance, whose whole bulk is 9 cubic inches, and which is $\frac{2}{3}$ as heavy as water, must float with 6 cubic inches immersed; for 6 cubic inches of water will be as heavy as 9 cubic inches of wood. And so generally,

$$\frac{\textbf{immersed volume of a floating body}}{\textbf{whole volume}} = \frac{\textbf{weight of unit vol. of solid}}{\textbf{weight of unit vol. of liquid}}$$

= relative specific gravity of solid.

Since ice, for instance, has a specific gravity of $\frac{11}{12}$, that is, since 12 cubic feet of ice weigh the same as 11 cubic feet of water, it follows that an iceberg must have $\frac{11}{12}$ of its whole bulk im-

mersed; hence the visible berg is only $\frac{1}{12}$ of the whole mass, there being eleven times as much underneath the water. So also a floating cork whose specific gravity is $\frac{1}{4}$ has $\frac{3}{4}$ of its volume projecting above the water.

Determination of Densities.

182. The foregoing principles are all remarkably well illustrated by their practical application to the determination of density or specific gravity.

The density of a body is defined as its mass per unit volume, or the mass of any volume divided by that volume,

$$\rho=\frac{m}{v};$$

similarly we might define specific gravity as the *weight* per unit volume, or the weight of any volume divided by that volume,

$$s=\frac{w}{v};$$

which would make specific gravity be to density as weight is to mass; or, as weight is g times mass (sect. 64), the specific gravity of a substance would be g times its density.

This, however, is not the definition of the term specific gravity as ordinarily used; it is the definition of what is called *absolute* specific gravity, which for distinction has been here called 'specific weight,' whereas the ordinary or *relative* specific gravity is the weight of any volume of a substance compared with the weight of an equal volume of some standard substance. The relative specific gravity of mercury with reference to water, for instance, is 13·6; of wood is, say ·6, and so on.

When one speaks of *the* relative specific gravity of any body, without stating the standard substance to which reference is made, it is understood that that standard substance is water; and so we may define *the* relative specific gravity, or *the* specific gravity of a substance, as the weight of any volume of it divided by the weight of the same volume of water.

Its relative *density* is precisely the same thing, both being simple numbers of equal value, but one having a direct reference to weight, the other to mass. In all comparative methods of

measurement we need make no distinction between density and specific gravity.

We have, in the preceding chapter, used s as standing for the absolute specific gravity or 'specific weight' of substances in general, m for that of mercury, and w for that of water; so the relative specific gravities of the three things are, of course, $\frac{s}{w}$, $\frac{m}{w}$, and $\frac{w}{w}$; the relative specific gravity of water itself being, of course, unity.

In the C.G.S. system of weights and measures, the absolute and relative specific gravity of a thing are represented by the same number, because the unit volume of water is defined to be the unit of mass (cf. sect. 171). The absolute specific gravity of water, or the weight of 1 cubic centimetre, is 1 gramme; and if a thing is three times as heavy as water, a cubic centimetre of it weighs 3 grammes, and so on.

To compare the Specific Gravities of two Liquids.

183. *1st Method.*—If they do not mix, place them one in each of the two legs of a U tube, and measure the heights of their respective columns (sect. 176); then $\frac{s_1}{s_2} = \frac{h_2}{h_1}$.

This method is used sometimes to compare with great accuracy the relative densities of a liquid at different temperatures. (Dulong and Petit's method for absolute coefficient of expansion of mercury; see Ganot, art. 273, or Deschanel, Part II., page 287.) A convenient modification is to dip the open ends of an inverted U tube into the two liquids, each in its own beaker, and to suck air out of the bend of the tube by means of a T piece, until the liquids rise and stand as measurable columns.

2nd Method.—Weigh a bottle full of the first liquid, and then the same bottle full of the second; deduct from each the weight of the bottle, and you will have the weight of the same volume of the two liquids to compare. In symbols, if b is the weight of the empty bottle,

$$\frac{s_1}{s_2} = \frac{w_1 - b}{w_2 - b}.$$

This method is often used. Flasks ('specific-gravity bottles') are made for the purpose (fig. 85). They are very light, and are arranged so that they can be accurately filled always to the same extent. For this purpose their neck has a constriction with a ring drawn round it with a diamond, and they are always filled up to this ring. This is done by filling them at first too full and then extracting the surplus with a scrap of blotting-paper or a capillary tube. The stopper is then inserted to prevent evaporation, and the whole is weighed in a delicate balance. The weight of the empty bottle, b, must have been previously ascertained.

Fig. 85.

3rd Method.—Take any non-porous solid heavier than both liquids and insoluble in either of them, such as glass, weigh it

first in air (or vacuum), then immerse it wholly in one of the liquids (hanging it from the pan of the balance by a fine wire or hair), and weigh it in that. It now weighs less by the weight of the liquid it displaces—note this loss of weight. Now weigh it in the other liquid, and note its loss of weight in that. The same volume of each liquid has been displaced, and the first loss of weight was the weight of this volume of the first liquid; the second loss, the weight of the same volume of the second liquid; so the specific gravity of the first liquid referred to the second, is the ratio of the first loss to the second loss. Or in symbols, if w is the weight of the solid in air, and w_1 and w_2 its weight in the two liquids respectively,

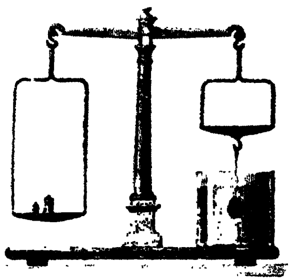


Fig. 86.—Hydrostatic Balance.

$$\frac{s_1}{s_2} = \frac{w - w_1}{w - w_2}.$$

This method has been used by Matthiessen to determine the expansibility of water by heat, and it is called the areometric method. (See Balfour Stewart's *Heat*, page 51.)

The operation of weighing a solid under a liquid is conducted by an ordinary balance with one of its pans replaced by a much shorter one with a hook under it, to which the solid can be hung by a fine platinum wire (fig. 86). When so arranged, it is often called a hydrostatic balance.

4th Method.—Take an insoluble and non-porous solid lighter than all the liquids you have to compare, and float it in each of them; ascertaining in each case the volume of it immersed. The weight of this volume of the liquid must in each case be equal to the weight of the solid, which is constant; so we obtain a set of different volumes all of the same weight. Call these volumes v_1, v_2, v_3, &c., and let w be the weight of the solid; then,

$$\text{since} \qquad w = v_1 s_1 = v_2 s_2 = v_3 s_3 = \&c.,$$

the ratios of the specific gravities to one another are inversely as the immersed volumes. Instruments for carrying this out are made of glass or metal, and sold under the name of *hydrometers* (see sect. 186).

5th Method.—Take a solid lighter than all the liquids, and

float it in each, loading it so as to immerse the same volume in all; that is, always make it sink to a fixed mark. The weight of this volume of the liquid is the weight of the solid plus the load, so the specific gravities of the liquids are as the numbers representing this total weight in the different cases.

An instrument for carrying this out is called Fahrenheit's hydrometer, but it is seldom now used.

Another method is given in Ganot, art. 121.

To determine the absolute Density of a Liquid.

184. 1*st Method.*—Weigh a known volume of the liquid in a gauged specific-gravity bottle (fig. 85), and divide the mass by the volume.

2*nd Method.*—Weigh a solid of known volume before and after immersion in the liquid, say a sphere of measured diameter. Its loss of weight will be the weight of its own volume of the liquid, so the weight of unit volume of the liquid is $\frac{\text{loss of weight of solid}}{\text{volume of solid}}$.

To determine the absolute Density of a Solid.

Weigh a known volume of it, say a sphere or a cube or something easily gauged, and divide the weight by the volume.

To compare the Densities of a Solid and a Liquid.

185. 1*st Method.*—If the solid be heavier than the liquid. Weigh it in air and in the liquid, and divide the weight in air by the loss of weight in the liquid; the quotient is the relative specific gravity of the solid referred to the liquid; $s = \frac{w}{w - w'}$.

2*nd Method.*—Applicable only if the solid be lighter than the liquid. Float it in the liquid, and take the ratio of the volume immersed to the whole volume (sect. 181).

If the solid is a cylinder floating upright, volumes are proportional to lengths; and the specific gravity is then $\frac{\text{length immersed}}{\text{whole length}}$.

3*rd Method.*—If the solid be lighter than the liquid. Weigh it first in air; then immerse it in the liquid by attaching a heavy body to it to sink it, and weigh the two together. Also weigh the sinker by itself in air and in the liquid. The loss of weight

of the two together gives the weight of liquid displaced by both; the loss of weight of the sinker alone gives the weight of liquid it displaces; therefore the difference of the two losses gives the weight of the liquid displaced by the body itself—that is, the weight of an equal volume of the liquid. So the relative specific gravity of the solid is its weight in air divided by the difference of the two losses.

If the body is porous like coke or pumice, it is necessary to distinguish between its average density, including pores, and the actual density of its material. In the former case it may be varnished over; in the latter case it is best to pound it in a mortar, and treat it by the method immediately to be described.

A liquid must always be chosen in which the solid is not soluble. Thus, for a piece of rock-salt, one must not use water, but either some such liquid as turpentine or benzol, or a saturated solution of salt; and the specific gravity of the salt referred to this liquid must be multiplied by the specific gravity of the liquid to give the specific gravity of the solid with reference to water.

Another, though essentially similar, method is given under Nicholson's hydrometer, sect. 186, which see.

4th Method.—Useful when the solid is porous or in the form of a powder. The difficulty with a powder is that it is impossible to gauge the volume of the solid particles directly, and also difficult to suspend the powder in water so as to determine its loss of weight. A specific-gravity bottle with a wider neck than that shown in fig. 85 is used. Ascertain the weight of the bottle when empty, and also the weight of water it will contain when full up to the mark. Put a known weight of the powder into the bottle, and fill up with water; the powder displaces some water, so it will not now hold so much as before the powder was in; but the weight of the whole, minus the weight of the powder and bottle, gives the weight of the water now in. The difference between this weight and the weight of water the empty bottle originally contained, gives the weight of water displaced by the solid powder; so the specific gravity of the solid is

$$\frac{\text{weight of powder}}{\begin{array}{c}\text{weight of water required to}\\ \text{fill empty bottle}\end{array} - \begin{array}{c}\text{weight of water required to fill up}\\ \text{bottle after the powder is in.}\end{array}}$$

If the powder be soluble in water, of course some other liquid must be used: the result can be multiplied by the specific gravity of this liquid, if the specific gravity of the powder referred to water be required.

5th Method.—A more elaborate method, also serviceable when

the solid is soluble or porous, is to use it to displace only *air*, in an arrangement something like Boyle's tube (fig. 103), which is called a volumenometer.

Hydrometers.

186. A hydrometer is a light body loaded so as to float in stable equilibrium at the surface of a liquid, and of a shape which renders it easy to observe accurately how much of its volume is immersed; and its use is to compare the specific gravities of liquids, or of solids and liquids. See methods 4 and 5, sect. 183, and methods 2 and 3, sect. 185. They are of two classes.

1st, Hydrometers of variable immersion or common hydrometers (Twaddell's, Beaumé's, Sykes', &c.).

2nd, Hydrometers of constant immersion (Nicholson's and Fahrenheit's).

1st Class.—Common hydrometers are glass cylinders or 'stems,' loaded and arranged so as to float upright. This is done by making them terminate below in a couple of bulbs, one full of air, the other full of mercury or shot (fig. 87). They must be of such weight as to float in a liquid with part of the cylindrical stem projecting; hence they are usually sold in sets, say a set of three, one for heavy liquids, one for medium, and one for light. The heavier the liquid the more of the stem projects, but in a light liquid they sink pretty deep—always sinking until they have displaced their own weight of the liquid. A thin stem makes the instrument sensitive, a wide stem diminishes its sensitiveness, but increases its range.

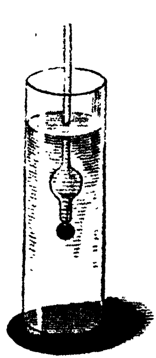


Fig. 87.

The specific gravity of the liquid is (see sect. 181),

$$\frac{\text{the whole volume of the instrument}}{\text{the volume immersed}} \times \frac{\text{the weight of the instrument}}{\text{the weight of an equal volume of water}}$$

(the last fraction being the average specific gravity of the instrument); that is, the specific gravity of the liquid varies inversely with the volume immersed. The stem, however, is graduated so that the specific gravity is read off directly from the numbers on it.

2nd Class.—Hydrometers of constant immersion will serve not only to compare the specific gravity of liquids, but also to deter-

mine the specific gravity of any solid, whether heavier or lighter than water, and this is their principal use; they will, moreover, make a very good substitute for a common balance. They consist, like the others, of a floating cylinder or rod, which, however, is usually made very thin (often only a wire), and instead of being graduated, has one fixed mark on it, to which it is always sunk. Its appendages are, a tray, A; a large light bulb, B; and a heavy bulb, or tray and cage, C. Fahrenheit's has only a shotted bulb below, and is made of glass. Nicholson's is made of metal, so that it cannot be used in corrosive liquids. It is, in fact, only used floating in water to determine the specific gravity of solids: it is the one which has the tray and cage C, and is shown in fig. 88.

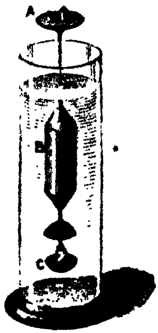


Fig. 88. Nicholson's Hydrometer.

It will not sink down to the fixed mark *m* on the fine cylindrical stem, unless some extra weights are put on the tray A; let 20 grammes be the weight required to sink the instrument by itself. This constitutes a constant of the instrument which must be known or determined before use.

Now to use it as a common balance, you place on the upper tray the body you wish to weigh, and then add weights, say $6\frac{1}{2}$ grammes, till it has sunk to *m*; one then knows that the body weighs $20 - 6\frac{1}{2} = 13\frac{1}{2}$ grammes.

To use it as a hydrostatic balance, you place the body in the lower tray; and now it requires say 3 more grammes to sink it to the mark, showing that the solid has lost 3 grammes of weight by being immersed in water, hence this is the weight of the water it displaces; and its specific gravity is therefore $\frac{13\frac{1}{2}}{3} = 4\frac{1}{2}$ (cf. method 3, sect. 185).

(Its weight when under water is, of course, $10\frac{1}{2}$ grammes.)

Suppose the solid had been lighter than water, and that when it was in the upper tray 12 grammes had been required to sink the instrument, whereas, when placed in the lower tray (where, of course, it would tend to float upward, and have to be confined by the cage), 30 grammes were required; then the loss of weight in water would be 18 grammes, and as its weight was 8, its specific gravity would be $\frac{4}{9}$.

(Its weight when under water is -10 grammes; that is, 10 grammes upwards.)

Equilibrium of Floating Bodies as regards Rotation.

187. We have now learned that a body necessarily floats in a liquid whenever it displaces its own weight of that liquid—that is, that under these circumstances the two contrary forces, its own weight and the resultant of all the fluid pressures on its surface, are equal, and are hence in equilibrium as far as *translation* is concerned. But in order that there may be also equilibrium as regards *rotation*, these two equal contrary forces must act along the same straight line; in other words, since the weight of the body passes through its centre of gravity, the resultant of the fluid pressures must also pass through this point; or, again, in other words, the centre of pressure (sect. 177) of the immersed surface must lie vertically under the centre of gravity of the body.

When this condition is satisfied there is complete equilibrium; but there remains the question whether this equilibrium is stable or not.

It is manifestly stable if the point of application of the upward force is above the point of application of the downward one.

Now, just as the downward force, the weight of the solid, may be considered as acting at its centre of gravity, so the upward force, the weight of the liquid displaced, may be considered as acting at *its* centre of gravity; and this point, the centre of gravity of the liquid displaced, is the real *centre of buoyancy or flotation;* the term 'centre of pressure' being commonly applied only to simple surfaces which displace no water. The centre of pressure is always a point on the *surface*—namely, that point where the line of resultant pressure meets the surface. This line of resultant pressure, which is vertical, and which always passes through both the centre of pressure and the centre of buoyancy, may be called the *line of buoyancy.*

If, then, the centre of gravity of the water displaced be above the centre of gravity of the solid, the equilibrium is certainly stable.

This, however, cannot be the case with *homogeneous* solids; it can only be satisfied by loading the floating body. And it is satisfied in all the above hydrometers; their centre of gravity being down near the shotted bulb, while the centre of gravity of the water displaced is up near the centre of the air-bulb; consequently their equilibrium is very stable.

But unless the floating body is totally immersed, it is quite possible to get stable equilibrium without satisfying the above condition; in other words, this condition is *sufficient*, but not *necessary*, for bodies floating at the surface of a liquid.

For instance, in a canoe, the joint centre of gravity of canoe and occupant is much higher than that of the water displaced by it; and so it is in ships and boats generally, though ballast is used to keep the centre of gravity of a vessel reasonably low.

The higher the centre of gravity of a vessel is, the less is its stability; and by making it high enough, the equilibrium is sure to become unstable, so that the least disturbance will cause the body to rotate or turn over into some more stable position.

You will find an example of unstable equilibrium if you try to float an empty bottle or a common pencil upright. A penholder, however, or a bottle half full, will float upright one way, because loaded.

A long cylinder like a pencil or wine-cork floats in stable equilibrium on its side; but a short cylinder like a flat plate or a collar-box will float with its length vertical. A sphere rests in neutral equilibrium in any position; and so does a totally immersed homogeneous body of the proper weight, whatever may be its shape.

188. To investigate fully the conditions of stability or instability of equilibrium, it is no use taking the body just in its position of equilibrium with the two equal forces acting along the same vertical line, any more than it was when a round-based body was standing on a flat table (in sect. 144); but one must imagine the body tilted a little, so that the equal forces act along different though parallel lines—that is, form a couple—and observe whether the effect of this couple is such as to restore the body to its original position, or whether it tends to increase the displacement more and more. In the former case the equili-

brium in the original position was stable; in the latter, it was unstable.

Let 1 (fig. 89) be a hemisphere floating in water in equilibrium, and therefore with the two centres of gravity, G of the body, and C of the displaced water, in the same vertical line—the line of buoyancy or resultant pressure. And let 2 be the same body disturbed from equilibrium into a new position, and therefore with a new centre of buoyancy, C_1. We have then the downward force w acting at G, and the upward force equal to w acting at C_1; the two constituting a couple of moment $w \times ab$, whose tendency is to restore the body into its original position; which was therefore one of *stable* equilibrium. In 3, this fig. 2 is repeated, but the old centre of buoyancy, C, of fig. 1 is indicated in the body as well as the new one, C_1; and the old line of buoyancy, CG, is produced till it cuts the new one through C_1 in the point M; which, in the case supposed, happens to be the centre of the sphere.

This point M is the *metacentre;* as already explained in connection with rolling bodies in sect. 146.

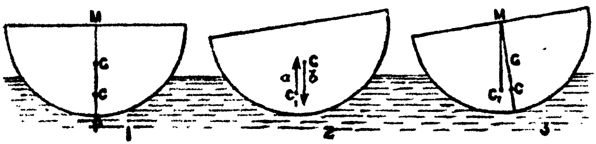


Fig. 89.

The *metacentre* is defined as the intersection of the old line of buoyancy, drawn in the body when in equilibrium, with the new line of buoyancy when the body is slightly disturbed from its position of equilibrium; and the rule for stability is:

If the metacentre M is above the centre of gravity G, the equilibrium is stable.

If it is below G, the equilibrium is unstable.

If M coincides with G, it is neutral.

And the height of M above G measures the *stability*.

All this will be seen at once if one just considers the couple as in fig. 2 above. For consider the upward force acting through the point M on the line GC fixed to the body (fig. 3); if M is above G, the upthrust will tend to restore the body and to bring GC upright again, the moment of the couple being proportional to the length MG; whereas, if M is below G, it tends to topple the body over more and to turn the line GC more and more from the vertical.

The position of M depends on that of the new centre of buoy-

ancy, and this depends on the shape of the floating body about the water-line. The shape of a ship or boat is devised so as to make the metacentre as high as possible, see fig. 90.

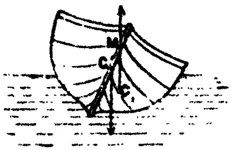


Fig. 90.

Strictly speaking, the disturbance from equilibrium ought to be infinitely small in order to give the correct position of M, and the correct measure of the stability MG. If the disturbance be great, the metacentre will in general be in a different position. If a ship lurches too much, the metacentre comes down very low, and may even pass below G; in which case, unless all the men can rush to one side, so as to alter the position of G, or unless an opportune wave comes to right the vessel, it must heel over, like the *Captain*.

Thus, in a floating body in equilibrium, there are four points vertically over one another (see fig. 89, No. 1):

M, the metacentre;
G, the centre of gravity of the floating body;
C, the centre of gravity of the fluid displaced; and
P, the centre of pressure of the immersed surface.

Of these P is always the lowest; and M is always above C (hence if C happens to be above G, much more is M); and the stability or instability of the equilibrium depends on whether M is above or below G.

As a matter of fact, a ship, like many floating bodies, has two metacentres; one, the one ordinarily spoken of as *the* metacentre, concerned in rolling; the other, very high up and of no practical account, concerned in pitching. It would be next to impossible to upset a ship by tilting it at the bows. In the *circular* Russian ironclads the two coincide. In an ordinary wine-cork floating on its side, one metacentre, the rolling one, coincides with the centre of gravity of the cork; the other, the pitching one, is a good height up.

In bodies of irregular shape the two lines of buoyancy, CG and the vertical through C_1, need not intersect at all, for they may lie in different planes: such bodies have no metacentre at all.

The whole subject of the metacentre, however, is not one that can be treated in an elementary book like the present; and it will be sufficient to have indicated the sort of ideas connected with the stability of equilibrium of floating bodies.

EXAMPLES—XXII.

(1) Find the force with which a sphere one metre in radius is urged upward, if it is totally immersed in water.

N.B.—Observe that the *depth* to which it is immersed is now immaterial.

(2) Find the apparent weight of a decimetre cube of stone in water, if its specific gravity is 2·5.

(3) How much of this block of stone would project above the surface of mercury in which it was floating?

(4) A solid which weighs 35 grammes *in vacuo* weighs when immersed in water only 5 grammes, while in another liquid it weighs 14 grammes; find the specific gravity of this liquid.

(5) The stem of a common hydrometer is graduated into 100 equal parts. The bulb and immersed portions, when it is sunk to the division 0, are equal to 3 times the stem in bulk. If it sinks to 20 in water, what will be the specific gravity of liquids in which it sinks to 80 and to 0 respectively?

(6) How deep would the hydrometer of the last question sink in a liquid of specific gravity ·8?

(7) If a floating body projects $\frac{1}{6}$ of its bulk above water, what will be the specific gravity of a liquid from which $\frac{1}{3}$ of its bulk projects?

(8) If a centimetre cube of metal weighs 8·5 grammes under water, what is its true weight?

(9) A Nicholson hydrometer which will sink to the fixed mark if 20 grammes be placed on the upper tray, requires 5 grammes more if the weights are placed on the lower tray beneath the surface of the water instead of on the upper one. What is the specific gravity of the metal of which the weights are made?

(10) A body A weighing 3 grammes is attached to another body B weighing 6 grammes, and the whole immersed under water, when they are found to weigh 2 grammes. The body B alone weighs 4 grammes under water. What are the specific gravities of A and B?

(11) A specific-gravity bottle, when empty, weighs 15 grammes; when full of mercury, it weighs 151 grammes; and when full of another liquid, it weighs 33 grammes; what is the specific gravity of this liquid?

(12) The above bottle, when 8 grammes of a certain sand have been introduced, and the rest filled up with water, weighs altogether 30·5 grammes; what is the specific gravity of the sand?

(13) A piece of cork weighs 10 grammes. A piece of iron is attached to it, and the two together weigh in water 20

grammes. The iron alone in water weighs 70 grammes. Find the specific gravity of the cork.

(14) A body weighs 10 lb. in air and 8 lb. in water. Find its volume and specific gravity.

(15) Find the density of a piece of wood from these data: weight of wood = 230 grammes, weight of a piece of iron in water = 580 grammes, weight of the wood and iron together in water = 465 grammes.

(16) A specific-gravity bottle, completely full of water, weighs 38·4 grammes, and when 22·3 grammes of an insoluble solid powder have been introduced it weighs 49·8 grammes. Calculate the density of the solid.

(17) Find the specific gravity of sugar from the following data: a flask, which can just be filled with 50 grammes of alcohol, whose specific gravity is 0·8, has 20 grammes of sugar put into it, and is then filled up with alcohol, the contents then weighing 60 grammes.

(18) A cubical block of wood measures 8 inches in the side. It floats in water with four edges vertical, and with one inch above the water. Find the specific gravity of the wood.

(19) The specific gravity of a certain solution of sugar in water is 1·2. Find the weights of sugar and water in 100 parts of it, being given that the specific gravity of sugar is 1·5, and supposing that no change of volume occurs on making the solution.

(20) Find the length and specific gravity of a cylinder which floats in water with 2 inches of its vertical axis out of the water, and also in a liquid of specific gravity 1·5 with 6 inches out of it.

(21) Compare the apparent weights of equal masses of lead and cork, (1) in water; (2) in air of specific gravity $\frac{1}{800}$. [The specific gravities of lead and cork are 11 and $\frac{1}{4}$ respectively.]

(22) 3 oz. of sugar of specific gravity 1·5, are dissolved in 8 oz. of water. Find the specific gravity of the mixture (1) assuming that its volume is the sum of the volumes of its constituents, (2) assuming that the volume of the water is unchanged, (3) assuming that the volume of the solution is increased by half the bulk of the sugar.

(23) Find the specific gravity of naphtha if a piece of potassium of specific gravity 0·84 and weight 20 grammes in air weighs $3\frac{1}{4}$ grammes in naphtha.

(24) If equal weights of sugar and water result in a solution of

density 1·4, how much condensation of volume has occurred?

(25) A glass stopper weighs $2\frac{1}{4}$ oz. *in vacuo*, $1\frac{1}{2}$ oz. in water, and $1\frac{3}{4}$ oz. in spirit. Find the specific gravities of glass and spirit respectively.

(26) Determine the thickness of a metal wire, a piece of which, 3 metres long, weighs ·24 grammes in air and ·21 grammes in water. Find also the weight of a cubic centimetre of the metal.

(27) A flask, which when filled with water weighs altogether 410 grammes, has 80 grammes of a solid introduced, and being then filled up with water weighs 470 grammes. What is the volume of a kilogramme of the solid?

(28) A solid weighs 117 grammes in air, 98 in water, and 101 in another liquid. Calculate the specific gravities of the solid and the liquid.

(29) A piece of opaque paraffin wax (sp. gr. ·9) contains imbedded in it a sphere of glass (sp. gr. 2·5). The whole weighs 50 grammes in air and 20 grammes in water. How big is the sphere?

(30) How much silver is contained in a 'gold' crown if it weighs 985 grammes in air and 918 in cold water, taking gold as being 19 times and silver 8 times as heavy as water?

CHAPTER XIII.

ON THE PRESSURE OF THE ATMOSPHERE, AND ON THE PROPERTIES OF GASES.

(Pneumatics.)

189. Most of what we have said in the last two chapters about liquids is equally true of all fluids. Gases have the same mobility of particles, and therefore transmit pressure equally in all directions. Gases are subject to gravity, and therefore press upon all surfaces exposed to them with a pressure depending on their depth and density; and they exert a sustaining force on bulky bodies equal to the weight of the gas displaced by those bodies, thus causing them to lose weight, and if very light to float upwards; thus acting just like liquids. Hence, the only part of the two preceding chapters which does not apply to gases is that which relates directly or indirectly either to the constancy of density or to the free surface of a liquid—a free surface being precisely the thing which a perfect gas never has. It is infinitely expansible.

This and all other peculiarities of gases as distinguished from liquids are due to the fact that their elasticity of volume is not constant or dependent on the gas itself, but depends on the pressure to which at the time the gas happens to be subject; but all the special properties of gases, *qua gases*, we will reserve for consideration in sect. 198 *et seq.;* at present we will only deal with those properties which they possess in common with all fluids.

PART I.—THE PRESSURE OF THE ATMOSPHERE.

190. Now we live immersed in an ocean of air of unknown and indefinite depth, and hence we and all terrestrial surfaces experience its weight just as if it were an ocean of liquid; and many phenomena of common life depend upon this pressure. Its intensity may be expressed in pounds-weight per square inch, or grammes-weight per square centimetre, or units of force per unit area; it is not quite constant at any one place, varying with many apparently accidental and local circumstances, but its average value is 1033 grammes weight (or 981 times this number of dynes), per square centimetre, or 14·6 lb. weight per square inch, or roughly, a ton weight per square foot.

Hence, a man's body experiences a total pressure of about 18 tons weight, for we found his average cross section (sect. 174) to be 80 square inches, which is that of a rectangle $8'' \times 10''$, whose periphery is 3 feet; so, if the man be 6 feet high, his surface, without allowing much for irregularities, is 18 square feet.

The pressure is exerted with perfect uniformity on all sides, and not only on the outside but on the inside too, so that it is not felt. The only way to make it appreciated is to destroy its uniformity by partial removal. If the pressure be removed from one side of any surface, then the other side experiences the whole uncompensated pressure of $14\frac{1}{2}$ lb. per square inch. If the air be withdrawn from any closed vessel, the outside experiences a crushing pressure, and if not very strong it will collapse.

Again, if the air be removed from a vessel whose mouth is beneath the surface of a liquid, that liquid is forced up into the vessel by the atmospheric pressure on the rest of the surface, the weight of the air sustaining the weight of the liquid, and completely filling it if the vessel is not too high. The product sh, which expresses the intensity of pressure of the liquid (sect. 175) at the mouth of the vessel, must therefore be about 1033 grammes weight per square

centimetre, if the liquid is supported by the average pressure of the air. Now, if the liquid be water, s equals 1 gramme per cubic centimetre, consequently h cannot be much greater than 1033 centimetres (or about 34 feet); if the vessel were taller than this, it would not be full. The atmosphere can therefore support a column of water 34 feet high, but of mercury, which is 13·6 times as heavy as water, it can only support a column 76 centimetres (about 30 inches) high. (For note that $34 \times 12 = 30 \times 13{\cdot}6$, as it happens, exactly.)

Modes of Removing the Air from Vessels.

191. One way of exhausting a vessel is to drive out the air by steam, and then condense the steam.

Fig. 91.

Experiment 1.—Boil water in an air-tight tin canister and cork it up: remove the lamp and pour cold water over it: the uncompensated pressure outside will crush it.

Experiment 2.—Take a long tube closed at the top and bent as shown in fig. 91; fill it completely with steam, and dip its open end under mercury. As the steam condenses, the mercury is forced up to a height of nearly 30 inches, and the tube may then be removed from the basin of mercury and carried about. The weight of liquid in one limb of the tube is balanced by the weight of the atmosphere in the other, which may be supposed to be extended to the top of the atmosphere (compare fig. 83, Chap. XI.).

A still simpler way of removing air from a tube is to fill it with a liquid. This is the way in which Torricelli originally performed the experiment and measured the pressure of the atmosphere. He filled a long tube with mercury without air-bubbles, and then inverted it with its mouth under mercury in a basin (fig. 92). On removing his finger, he saw the mercury descend till its surface was 29 or 30 inches above that of the liquid in the basin, and there come to rest after a few oscillations.

Above the mercury was a nearly perfect vacuum, now called a Torricellian vacuum. If any gas or vapour be introduced into this it will depress the column more or less against the force of the atmosphere. For instance, the water vapour left in the cold tube after the experiment of fig. 91 will depress the column half an inch or so.

Pumps.—Another mode of removing air or any fluid from a vessel is by means of an arrangement of valves which open and permit egress one way only, combined with some method of squeezing the fluid so as to make it move in one direction or other. Such a combination is called a **pump**, and three kinds are shown in fig. 93. The valves in each are self-closing flaps (shown open in the figures for clearness), which will open upwards by pressure from beneath, but which only close more tightly if any pressure be exerted on them from above. (Such valves exist in the veins, and cause whatever flow there is to take place in one direction.) The compressing apparatus to cause motion in the fluid is in (1) an elastic bag to be alternately squeezed and relaxed by the hand—such an apparatus, without valves and open only at one end, is the lung of an animal; in (2) and (3) it is a piston fitting a cylinder which is to be pushed to and fro, or up and down; the peculiarity in (3) being that one valve is in the piston itself.

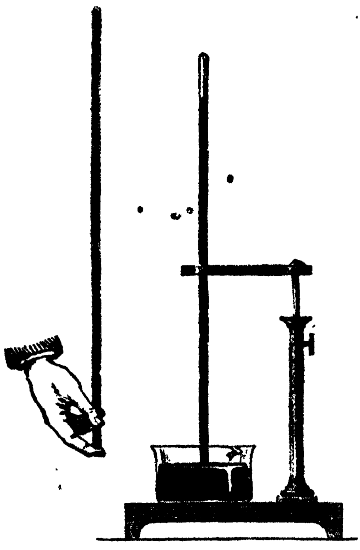
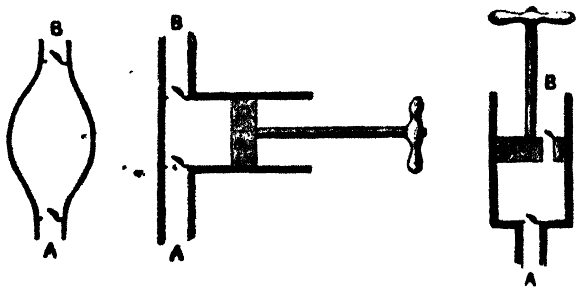


Fig. 92.—Torricelli's Experiment.

No. 1 is a pump used in surgery for producing injections or for delivering a strong jet of liquid. The heart of an animal acts on the same principle; so does a pair of blow-bellows imperfectly, for though it has only one valve, the narrowness of the nozzle acts partially as a second one.

No. 2 is a mere modification of No. 1, and is used in garden and fire engines. Both these are called **force-pumps.**



No. 1. No. 2. No. 3.

Fig. 93.

Pumps: with the valves displayed as if kept open by a wind from A to B.

No. 3 is called a lift-pump, because it gets the fluid above the piston and then lifts it up when the piston is raised. It used also to be called a 'suction' pump.

All three arrangements evidently tend to transfer any fluid they may contain from A to B, producing an exhaustion in any vessel screwed on to the end A, and a condensation in any vessel screwed on to B.

Modes of Measuring the Atmospheric Pressure.

192. **Aneroid Barometers.** — The pressure may be measured and its variations indicated by exhausting a

strong metal box with a thin and flexible (corrugated) top, supported by a spring against the weight of the atmosphere, as shown in fig. 94. If the atmospheric pressure increases, the spring is compressed a little more; if the pressure decreases, the spring recovers itself a little; and so the box lid indicates variations of pressure by moving in or out, and its motions may be magnified by a rack and pinion and long index as shown. Such an instrument is called a 'barometer' (weight-measurer), and being made without mercury this form of it is called 'aneroid.' The box must be empty, or its contents would vary in size with temperature, and so give spurious results. In so far as temperature affects the strength of the spring, the instrument has an error to be corrected or compensated.

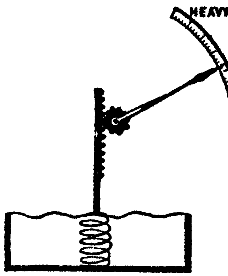


Fig. 94.
Skeleton diagram of the principle of an aneroid barometer, but in practice the spring is outside the box.

Mercury Barometers.—The mercury column (fig. 92) is a convenient measure of the pressure of the air, and is the original form of barometer. If the pressure increases, the column is forced higher up; if it decreases, the column descends.

It is found to oscillate on different days between 31 and 28 inches, at places near the sea-level; being usually high when the atmosphere over a country is quiet and dry, and the weather therefore fine and bright or frosty; whereas, when a portion of the atmosphere is in a state of whirling motion called a cyclone, the centrifugal force of the whirl causes a depression or region of low pressure near the middle of it, and the barometric column in such places is accordingly low. These large whirls of air travel along and convey with them much steamy warmth and clouds and rain, thereby destroying the continuance of fine bright weather and breaking up spells of frost. The approach of such

a cyclone, which, when violent, constitutes a gale or storm, is heralded by an incipient, and sometimes a rapid, fall of the barometer.

Fig. 95. Weather-glass.

Cyclonic or low barometer weather is characterised in Great Britain by warm, damp, muggy weather, with clouds and often rain and strong westerly winds. Anti-cyclonic or high barometer weather is characterised by calms or gentle dry winds, hot sun in summer, hard frost in winter, east winds in spring, and fogs in autumn.

These facts cause a barometer to be used as a weather-glass; and a convenient form, for popular purposes, is that of fig. 91, arranged as in fig. 95, where the motion of the mercury in the short open tube is used as the indicator instead of that in the long tube, and its motion is magnified by a float counterpoised over a pulley with an index; or else by a rack and pinion as in fig. 94. The advantage of this form is that the friction prevents very prompt motion, so that the accumulated changes of the last hour or two are indicated by the needle whenever you go and tap the instrument. As an accurate measurer of pressure, however, it is not of much use.

The cistern form (fig. 92) is always used for accuracy, and some mechanical arrangement is added by which the level of the mercury in the cistern can either be kept constant or can be read off; for, of course, when the mercury falls in the tube it rises in the cistern, and it is the *difference* of levels which really measures the pressure. Sometimes the scale on which the height of the column is read is adjustable, so that, before reading, its zero can be made to coincide with the level of the mercury in the cistern; by which device the required difference of level can be read off at once. If a barometer be carried up a mountain, the mercury column must descend, because some of the column of air which formerly balanced it is left below. By this decrease of atmospheric pressure, the

height of the mountain may be calculated. For more about barometers, see Deschanel or Ganot.

193. **Manometers.**—Columns of liquid may be used to measure pressures other than those of the atmosphere—such pressure gauges are called manometers. Fig. 96 shows a gauge for measuring the pressure of the steam in a boiler over and above that of the atmosphere by the height of a column of mercury; and the pressure may be stated as equal to so many inches or centimetres of mercury, or if very large, it may be stated as so many 'atmospheres'—every 30 inches of mercury being called one atmosphere.* Metal manometers are, however, preferred in practice.

Fig. 96.

By 'a pressure of 76 centimetres' on any area, then, is meant the pressure which would be produced by a column of mercury 76 centimetres high with that area as base. The intensity of pressure in grammes per square centimetre of a column of *water* is equal to its vertical height in centimetres (because 1 cubic centimetre of water weighs 1 gramme); or in absolute measure (dynes) its pressure is 981 times its height. That the pressure of a water column is numerically equal to its height, when expressed in gravitational C.G.S. units, is a convenient fact to remember. The pressure of any other liquid of specific gravity s is s times as great; so '76 centimetres of mercury' means a pressure of $76 \times 13{\cdot}6 \times 981$ dynes per square centimetre.*

Fig. 97.

The height of the column, of course, means the *vertical* height (cf. fig. 82);

* In the C.G.S. system of measures, a million dynes (or a megadyne) per square centimetre is conveniently called an 'atmosphere.' It is very nearly equal to 75 centimetres of mercury. Regnault unfortunately employed 76 centimetres as his standard pressure.

hence if a manometer or barometer tube be inclined, the mercury will flow further up the tube, but so that the vertical height of its surface is the same as before (fig. 97).

Modes of Raising Water.

194. The most obvious mode of raising water is to get something underneath it, and lift it up. This is the old method of a bucket and windlass. Since, however, the atmosphere can support a column of water about 34 feet high, it may be used to force water up from wells not much more than 30 feet deep. For this purpose, a tube is let down into a well, and then exhausted of air, either by filling it with steam and condensing it (which is nearly the oldest form of steam-engine or steam-pump, and was set up by Captain Savery at the water-works, York Buildings, Charing Cross, and used from 1698 to 1706, and in principle is still used in the modern 'pulsometer'), or by screwing the end A of one of the pumps of fig. 93 on to the tube, and working the pump. Fig. 98 shows pump No. 3 so applied, and is a common house-pump. First the air, and then the water, is transferred from A to B, and the water finds egress at the spout.

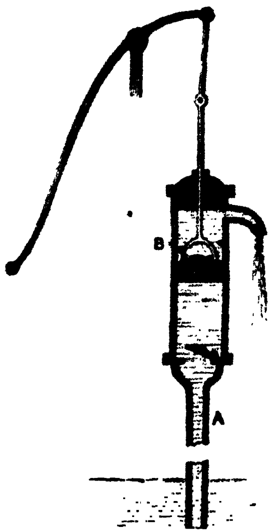


Fig. 98.—House-pump.

It is often required to raise water from mines several hundred yards deep. Atmospheric pressure is of course quite incompetent to effect this: the only plan is to get something under the water and lift it. Pump No. 3 is still used, only it is arranged at the bottom of the mine, within

20 or 30 feet of the water, and its spout is transferred higher up, so that it delivers the water at the top of the shaft. Water may be thus raised any height whatever. Such pumps are called lift-pumps, and are usually worked by engines at the top of the shaft; long rods connecting the piston of the pump with the beam of the engine.

A house-pump can also be used to lift water up to a cistern on the top of the house. The piston-rod of such a lift-pump works through a water-tight stuffing-box, as in fig. 98, but the spout has a tap by which it can be closed when desired; and a pipe leads from the upper portion, B, of the pump-barrel to the cistern.

Force-pumps Nos. 1 and 2 (fig. 93) are not used to raise water from any depth, but to deliver a strong jet; and fig. 99 shows the arrangement in a garden-engine. The stream of water is rendered continuous instead of intermittent, either by an elastic bag, or by an air-chamber. C is the air-chamber which contains air compressed by the over-supply of water, so that, if the pump stops working, the jet continues for a few seconds, only gradually diminishing in strength as the compressed air expands.

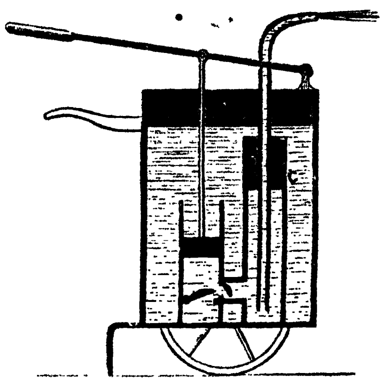


Fig. 99.—Garden Force-pump.

Fig. 76 showed a force-pump applied in the hydraulic press, with plungers instead of pistons. Plungers are indeed generally used in force-pumps; they act precisely like pistons of equal area, the only difference is that they fit the stuffing-box instead of the cylinder.

195. In any kind of lift-pump, the piston has, during its up-stroke, virtually to support a column of water reaching from the surface of the water in the well to the highest surface of water in the pipe. Calling this height h, and the area of the piston A, the pressure on it is wAh. To work the pump, a force somewhat greater than this must therefore be applied to the piston. In force-pumps, the pressure during the up-stroke corresponds to a column of water from piston to well; and during the down-stroke to a column from the piston to the highest point reached by the water, whether it be a free jet or confined in a tube (neglecting the friction of the moving water in all cases).

Mode of Lowering Water.

196. The force of gravity renders the lowering of water a very easy matter. If we have a liquid in a vessel, and wish to transfer it to another at a lower level, all that is needed is two holes in the vessel—one to let the liquid out, which must be below the surface, and the other to let the air in, which is best above the surface of the liquid; if it is beneath the surface, it may act, but it will do so irregularly, letting the air in by bubbles. One hole half beneath and half above the surface will act as two holes, and this is the way one empties a jug or bottle, rotating it till its one hole occupies this position. If the hole be large, it will act as two even if wholly beneath the surface, but the flow will be very irregular. The beer in a cask with the tap open, but without a venthole, is kept in by the atmospheric pressure, unless it is fermenting and forcing itself out by means of its own gas, or unless you blow up the tap. A pipette (fig. 100) is a vessel with two

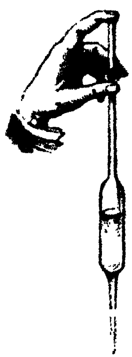


Fig. 100.—Pipette.

holes, and the flow of liquid from it can be stopped by closing either of them with the finger.

Siphon.—In an open glass vessel, however, it is not convenient to bore a hole through the glass beneath the surface of the liquid, neither is it always convenient to rotate the vessel till part of its mouth is below the surface. In such cases the necessary second hole may be introduced beneath the surface as one end A of a bent tube, whose other end, B, is at a lower level—say is immersed in another vessel at a lower level (fig. 101). If this tube be once exhausted of air, either by sucking liquid into it with the mouth, or by filling it at a tap before inverting it, the atmospheric pressure will afterwards keep it full of water; and the column of water in one leg, being longer than that in the other, will overbalance it, and a steady flow from A to B will be kept up till either the water sinks below the opening A, or till the level in both vessels is the same. Such a tube is called a 'siphon.' Its shape is wholly immaterial, provided that no part of it is at a height above the surface in either vessel greater than the column of liquid which the atmosphere can support, otherwise the action will cease. So also it would cease if it were put under the receiver of an air-pump and the air exhausted.*

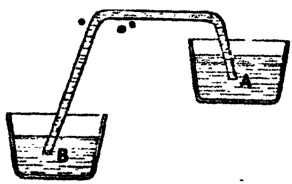


Fig. 101.—Siphon.

While the air was being exhausted, the flow would go on *with undiminished speed* until the air pressure became too weak to sustain the longer of the two columns; the liquid would soon then

* It is probable that, in a perfect vacuum, a siphon of moderate height would work perfectly well, because the cohesion of water free from air is pretty strong, and might maintain the continuity of the column of liquid in spite of gravity. Under these circumstances, the cause of the flow would be exactly like that of a chain over a pulley with one end longer than the other; and the analogy will be complete if the chain be supposed to uncoil itself from a table, and to coil itself up on the floor.

snap at the highest point, and the longer column would fall till it was the same length as the other. As the air pressure still further diminished, the two columns would slowly sink, like barometers, until, when there was no pressure left, the level of the liquid inside and outside the tube would be the same. On readmitting the air, the action would commence again, unless either end A or B was not fully submerged.

Fig. 102.

The shape of the siphon tube being immaterial, it might pass straight through the wall of the vessel from A to B (fig. 102), and such a pipe would empty the vessel to just the same extent, and at the same rate, as the tube of fig. 101 ; only it does not obviate the necessity of a hole through the side of the vessel as the tube bent over the edge does; neither, of course, would it cease to act in a vacuum.

Floating of Bodies in Air.

197. All things which displace any air (that is, which have any bulk) are pressed or buoyed up with a force equal to the weight of the air whose place they occupy (sect. 179), and so everything weighs less in air than it would in a vacuum. The true weight of a thing is its weight *in vacuo*, and this equals its apparent weight plus the weight of an equal bulk of air. The bulkier a thing is, the more does its apparent weight differ from its true ; and if a very light body be also very large, it may have no apparent weight at all, but may float about in equilibrium, or even be forced upwards, like a balloon.

What is called a pound of cork is therefore really more than a true pound, for it has been weighed against metal weights which are not so bulky as itself and displace much less air. A little demonstration-balance is sometimes made to hold a ball of cork and another of lead of the same apparent weight, so that they equilibrate each other in air; but if the buoyant power of the air be withdrawn by putting the whole under an air-pump, the cork will descend, showing that it is really the heavier of the two.

A thin copper or glass sphere with a tap may be used to measure this buoyant power. When the tap is open, very little air is displaced by the sphere; if you weigh it then, you get its true weight very nearly. But exhaust it and shut the tap. It now displaces a quantity of air, and accordingly is buoyed upwards, and will be found to be apparently lighter than before. The difference between its true and apparent weights gives the weight of an equal volume of air.

In this way 1 cubic centimetre of ordinary air, when the barometer stands 76 centimetres high and the thermometer stands at zero centigrade, is found to weigh ·001293 gramme. (This number ·001293 is therefore the sp. gr. of air referred to water; it is approximately equal to $\frac{1}{800}$.) Another mode of quoting the same result is to say that 11·2 litres weigh about 14·4 grammes; or 1 cubic inch weighs ·31 grain; or that a cubic foot weighs about an ounce and a quarter. Hence, since 1 gallon of water weighs 10 pounds, 1 pound of common air occupies about 80 gallons.

A sphere of brass a yard in diameter displaces, if exhausted, rather more than half a cubic yard of air, say 14 cubic feet, which weighs 17½ ounces about. If then its own weight were only a pound or so, it would ascend slowly like a balloon. But if so light as this, its walls could not be strong or regular enough to resist the pressure, and it would collapse. Such balloons are therefore impracticable. To sustain very thin walls against the air pressure, it is necessary to fill the balloon with some gas; and hydrogen, being the lightest gas known, is always used. Hydrogen enough to fill the above sphere would weigh only 1¼ ounce, so it would not add very greatly to the weight, and its presence enables the walls to be of thin oil-silk instead of metal. The first balloons were filled with hot air, which occupies more room and therefore displaces more than its own weight of cold air (see Deschanel, chap. xxi., or Ganot, art. 169).

EXAMPLES—XXIII.

(1) What is the height of the mercury barometer when the intensity of the atmospheric pressure is a megadyne

per square centimetre? (A million dynes is called a megadyne.)

(2) If a mercury barometer falls one inch, what will be the fall of a water barometer?

(3) Show that the oscillation of the column in a 'siphon' barometer, with its long and short limbs of equal cross section, is only half that of the column of a cistern barometer with an infinitely large cistern.

(4) Show that the motion of the top of the mercury in a barometer may be doubled by inclining the upper part of the tube at an angle of 30° to the horizon.

(5) What is the total pressure inside a steam boiler when the mercury gauge (fig. 96) stands at 150 centimetres and the barometer at 75?

(6) The piston of a lift-pump is 7 inches in diameter, and the depth of the water in the mine below the spout where the water is discharged is 533 yards. Find the least force which can raise the piston?

(7) If a rectangular mass of cork, dimensions $10 \times 8 \times 5$ centimetres, is counterpoised in air by 80 grammes of platinum, find the mass of the cork (neglecting the floating power of the air on the platinum).

(8) A mass of wood (sp. gr. ·6) is counterpoised by 105 correct grammes of iron (sp. gr. 7·5); find the mass of the wood (or its true weight *in vacuo*).

Ans. The volume of the iron is 14 c. c., so its apparent weight is $105 - (14 \times {\cdot}001293)$; and this is equal to the apparent weight of the wood, which is $x - (\frac{5}{3}x \times {\cdot}001293)$, where x is the number of grammes of the wood; hence $x = 105{\cdot}208$.

(9) A piece of metal weighs 2·4 grammes in mercury and 9 grammes in water; what would be its weight *in vacuo?*

(10) A lift-pump is used to lift water from a well whose water surface is initially 10 feet below the level of the pump to a cistern 18 feet above the pump-level. The diameter of the well is 3 feet, and the internal dimensions of the cistern are 4 feet long by 3 feet broad by 2 feet high. Find the work needed to fill the cistern.

The water in the well will be lowered x feet, where

$$\tfrac{1}{4}\pi(3)^2 \cdot x = 3 \times 4 \times 2$$

$$\therefore x = \frac{4}{\pi} \times \frac{8}{3} = \frac{32 \times 7}{66} = \frac{224}{66} = \frac{10}{3} + \frac{2}{33}$$

$$\therefore \text{ height of final lift} = 28 + \tfrac{10}{3} + \tfrac{2}{33} + 2$$

$$\text{and height of average lift} = 28 + \tfrac{5}{3} + \tfrac{1}{33} + 1 = 30 + \tfrac{2}{3} + \tfrac{1}{33}$$

$$\therefore \text{ work} = 3 \times 4 \times 2 \times w\,(30 + \tfrac{2}{3} + \tfrac{1}{33})$$

$$= 24 \times 62{\cdot}3\,(30 + \tfrac{2}{3} + \tfrac{1}{33}) \text{ foot lb.}$$

$$\begin{array}{rrl}
 & 62{\cdot}3 & \\
 & 24 & \\
\hline
 & 1246 & \\
 & 249{\cdot}2 & \\
\hline
 & 1495{\cdot}2 & (30 + \tfrac{2}{3} + \tfrac{1}{33}) \\
\hline
30 & 44856 & \\
\tfrac{1}{3} & 498{\cdot}4 & \\
\tfrac{1}{3} & 498{\cdot}4 & \\
\tfrac{1}{33} & 45{\cdot}3 & \\
\hline
 & 45898 &
\end{array}$$

Ans. 45900 foot pounds or about 20 foot tons.

(11) Determine the pressure of the atmosphere in lb. weight per square inch, correct to one decimal place, when the barometer stands at 29 inches.

(12) Find the greatest height to which oil whose density is 0·9 grammes per cubic centimetre can be raised by a common 'suction' pump, when the atmospheric pressure is a million dynes per square centimetre.

(13) On a day when the barometer stands at 76 cm., find the pressure in grammes per square cm. at a point 3 metres below the surface of a pond covered with a thin film of ice.

(14) A mercury barometer stands at 30 inches; find what it ought to read if it were sunk 50 feet below the surface of water.

(15) Find the value in grammes weight, and in dynes, per square centimetre, of a pressure able to sustain a 75 cm. column of mercury, at 0° centigrade, when its specific gravity is 13·596.

(16) The density of mercury decreases 180 parts in a million for every degree rise of temperature, hence find the pressure corresponding to a metre column of mercury at 20° C.

PART II.—ON PROPERTIES PECULIAR TO GASES.

198. A perfect fluid whose elasticity of volume (see Chap. X., sects. 154, 158) is equal to the pressure upon

it, provided the temperature is constant, is called a *perfect gas*. Many actual gases—namely, those called permanent gases, very nearly satisfy this definition; and we have now to consider what properties a gas possesses in consequence of this peculiarity.

First of all, gases must be very compressible: any additional pressure produces a corresponding change of volume. The increase of pressure (sect. 175) is the stress; the ratio of the change of volume to the original volume is the strain (sect. 160). Let the original pressure be P, and the new pressure P′; then the stress is $P' - P$. Let the original volume of the gas be V, the new volume V′, then the strain is $\frac{V - V'}{V}$. Its elasticity, when in the compressed state, by definition (sect. 158), is $\frac{P' - P}{\frac{V - V'}{V}}$; and for gases this is now stated to equal the pressure on it when in that state—namely, P′.*

$$\text{Hence} \qquad \frac{P' - P}{V - V'} = \frac{P'}{V}, \quad \text{or} \quad PV = P'V',$$
$$\text{or} \quad P : P' :: V' : V;$$

or, in words, the volume of a given quantity of a perfect gas varies inversely with the pressure, other things being equal. If the pressure be doubled, the volume is halved; if the pressure be halved, the volume is doubled. This is called Boyle's law, and may be verified by the bent tube of fig. 103. Its short leg is closed, its long leg open. Mercury poured down the long leg confines some air in the short one and compresses it, the whole pressure on the air

* If the strain takes place very suddenly, the elasticity is greater than P, being 1·4 times P. This is because the temperature does *not* then remain constant—heat is generated by the compression which has not time to escape. We will suppose, however, that all our compressions and expansions take place slowly enough to allow the temperature of the gas to remain without change.

in the tube being that of the atmosphere plus that of the column of mercury in the tube.

If the mercury stands 30 inches higher in the long leg than in the short, the original volume of the air will be found to be halved: for the original pressure it sustained was one atmosphere, and now it is two. Another 30 inches of mercury will make it shrink into one-third its original bulk, and so on. Under ordinary atmospheric pressure, 14·4 grammes of air occupy 11·2 litres (see sect. 197); but under a pressure of two atmospheres they shrink to 5·6 litres.

The shortest statement of Boyle's law is that, *ceteris paribus*,

$$PV = \text{constant};$$

but remember that *cetera* must be *paria*; the temperature must not change, *neither must the quantity* (that is, mass) *of gas*.

One gramme of hydrogen under a pressure of 76 centimetres of mercury, and at 0° centigrade, occupies 11·2 litres, or 11,200 cubic centimetres. Hence the value of the above constant PV for 1 gramme of hydrogen is in absolute C.G.S. units (see sects. 190 and 193):

$$76 \times 13{\cdot}6 \times 981 \times 11{,}200 = 1135 \text{ million ergs.}$$

Fig. 103. Boyle's Tube.

Call this K. It is the same constant for 16 grammes of oxygen, 14 of nitrogen, 22 of carbonic anhydride, and so on. It varies only with the absolute temperature. For 5 grammes of hydrogen or 80 grammes oxygen, the constant is 5K; it is, in fact, proportional to the mass of a gas, but varies for different gases with their molecular weights. A better statement of Boyle's law is that the ratio of pressure to *density*, $\frac{P}{\delta}$, is constant; for this is independent of everything but the nature of the gas and the temperature. If the pressure of any gas is divided by its specific weight, *s* or $g\rho$, the resulting constant is called *the height of the homogeneous atmosphere* of that gas at the given temperature (see example xxiv. 2).

199. The density of a gas, therefore (the mass per unit volume, see sect. 35), is directly proportional to the pressure. One consequence of this is, that as one ascends in the atmosphere, the pressure does not decrease uniformly as in the case of a liquid, but it decreases, at first at a more rapid rate, and afterwards more slowly. At a height of only three miles, for instance, the intensity of pressure is half what it is at the sea-level. For the pressure decreases not only by reason of the elevation, but also by reason of the diminution of density accompanying the decrease of pressure. Both causes combine, and the pressure diminishes upwards in what is called *geometrical* instead of in *arithmetical* progression.

200. But just as no actual liquids are perfect, so no actual gas is a perfect gas. They all deviate slightly from Boyle's law; they are probably not infinitely expansible, and are certainly not infinitely compressible, for many of them, if squeezed very much, condense into liquids; and as they approach their condensing point, they deviate from Boyle's law a good deal, becoming more and more compressible. Oxygen, nitrogen, and argon have now all been liquefied in bulk, and the two latter have been frozen. Hydrogen has been momentarily liquefied as a mist, and the only gas that has so far resisted even momentary liquefaction is helium. Still, all these gases are, at ordinary pressures and temperatures, a very long way off their condensing points, and they obey Boyle's law with considerable accuracy. They cannot, indeed, be condensed by any amount of simple squeezing; they have to be cooled enormously as well.

Air-pumps.

201. Air-pumps differ in no respect from other pumps except in details of arrangement. Their peculiarity is that the vessels they are used to exhaust or to fill contain always

the same volume of fluid; its density and pressure, however, are diminished or increased to any extent.

Pump No. 3 (fig. 93) is generally used for exhaustion, but pump No. 2 can also be used, and it will at the same time produce condensation in any vessel screwed on to its end B. It is then called a condensing syringe. If it obtains its air from the atmosphere, the same mass of air will be injected at every stroke, and consequently the pressure in a vessel screwed on to B will increase by a fixed amount at each stroke, that is, it will increase in arithmetical progression.

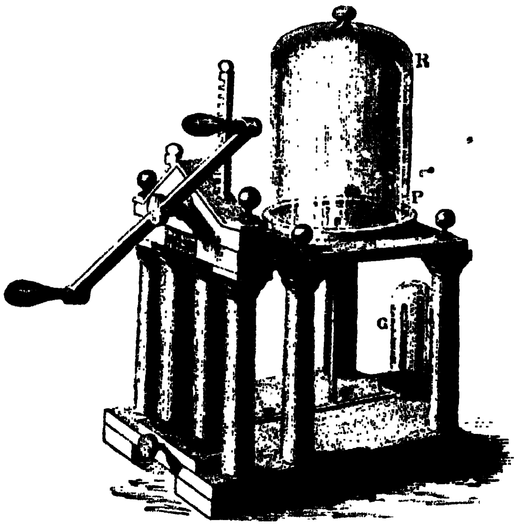


Fig. 104.—Air-pump.

Fig. 104 shows a double-barrelled air-pump with two of the No. 3 pumps arranged to exhaust a glass vessel known as the 'receiver.' At every stroke the air in the receiver expands to fill both receiver and pump-barrel, and the portion filling the latter is at the reverse stroke expelled into the atmosphere.

Call the volume of the receiver V, and that of a pump-

barrel v; the same *volume* of air, v, is extracted at every stroke, but not the same mass, because its density keeps on diminishing. If the pressure of the air in the receiver to start with, is P_0, and after the first stroke, P_1; the product of pressure and volume being constant, we have

$$P_0 V = P_1 (V + v).$$

The contents of the pump-barrel are now expelled, and the second stroke begins. During the second stroke the volume V again expands to fill the volume $V + v$, without the quantity of the air changing; so, if P_2 is the pressure after the second stroke,

$$P_1 V = P_2 (V + v).$$

Similarly, P_3, the pressure after the third stroke, is given by

$$P_2 V = P_3 (V + v);$$

and so on

The pressure after three strokes may therefore be written

$$P_3 = \left(\frac{V}{V + v}\right) P_2 = \left(\frac{V}{V + v}\right)^2 P_1 = \left(\frac{V}{V + v}\right)^3 P_0;$$

similarly, the pressure after n strokes is

$$P_n = \left(\frac{V}{V + v}\right)^n P_0.$$

The pressures $P_0\ P_1\ P_2\ P_3 \ldots\ldots P_n$ decrease, therefore, in a geometrical progression with the common ratio $\frac{V}{V + v}$.

Hence perfect exhaustion (or pressure equal zero) cannot be obtained, even with a perfect pump, without an infinite number of strokes.

202. To indicate the degree of exhaustion, a mercury gauge is commonly used, which may be simply a long tube reaching from the receiver into a cistern of mercury, something like a barometer; or it may be of the form shown

separately in fig. 105, and attached to the pump at G in fig. 104. The closed limb of the U tube is completely full of mercury, and remains so till the air pressure in the little bell jar which is exhausted with the receiver gets unable to support it; it then gradually descends as the exhaustion proceeds, and the pressure of the residual air in the receiver is measured by the difference of level between the mercury in the two limbs.

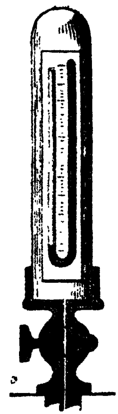


Fig. 105.
Air-pump Vacuum Gauge.

203. **Compressed-air Manometers.**—The diminution of the volume of a gas under pressure will measure that pressure in a more compact way than the mercury gauges of sect. 193 (compare the length of the two branches of the tube, fig. 103), and a manometer on this principle is shown in fig. 106. Faraday used to measure high pressures in his glass vessels by inserting little conical glass tubes, with one end sealed, containing air and a globule of mercury (fig. 107). As the pressure of the gas

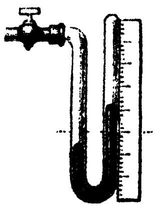


Fig. 106.

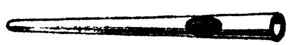


Fig. 107.

Simple compressed-air Manometers.

in which they were, increased, the globule moved up and compressed the air in the tube more and more; and the diminution of volume measured the increase of pressure, $P' = P\frac{V}{V'}$.

Lord Kelvin has applied the same principle to ocean

sounding, for, since every 34 feet of water adds another atmosphere to the pressure, if the pressure of the water be known, its depth can be calculated. A tube closed at one end is lowered into the sea, like a diving-bell, mouth downwards; and a registering arrangement records how far the water has entered the tube, and therefore how far the air in it has been compressed.

Diving-bells.—Open vessels containing air, then lowered into water mouth downward, need to be heavy in order to sink. It is usual to pump more air into a diving-bell as it sinks, so as to keep the water out notwithstanding its pressure; but if no such pumping be done, the original air will be compressed into smaller compass, the water will partly enter, and, since the displaced fluid is less, the buoyancy diminishes or the apparent weight increases as it gets more deeply immersed. On this principle little hollow glass figures, called 'Cartesian divers,' rise and sink in a liquid according as the pressure on them is diminished or increased.

EXAMPLES—XXIV.

(1) A barometer in a diving-bell indicates a pressure of 45 inches of mercury, the height of the barometer at the surface of the earth being 30 inches. What is the depth of the diving-bell?

(2) Find the height of the homogeneous atmosphere at zero centigrade. (This means the height an atmosphere must have, if it were made of incompressible fluid, of the same density as the real atmosphere at any point, and if it exerted the same pressure as the real atmosphere does at that point. See sect. 198.)

Ans. $76 \times 13{\cdot}6 \div {\cdot}001293$, in centimetres, about 8000 metres (or roughly about 5 miles).

N.B.—Notice that this does not vary with the barometric height.

(3) A siphon barometer which has a little air in its 'vacuum' indicates a pressure of only 72 centimetres; and on pouring

more mercury into the open limb until the vacuum is diminished to half its former bulk, the difference of levels becomes 70 centimetres. What is the true height of a proper barometer?

(4) The cylinder of an air-pump barrel has a capacity $\frac{1}{10}$th of that of the receiver it is used to exhaust. Find the pressure in the receiver after 1 and after 2 strokes of the pump, if the original pressure was 77 centimetres.

(5) If a quantity of air is squeezed in a closed tube into $\frac{1}{100}$th of its original volume which it occupied when the barometer was at 30 inches, what pressure does it indicate; and at what depth under water would this pressure be experienced?

(6) An inverted vessel like a diving-bell, 6 feet high and weighing half-a-ton, weighs apparently $\frac{1}{4}$ of a ton when lowered so that the water inside it is 86 feet below the surface. What would be its apparent weight when raised so as to be only just immersed, supposing that the same quantity of air now quite fills it?

(7) A Y tube is inverted, and each leg is dipped into a different liquid. Air is then sucked from its stem, and the liquids rise a vertical height of 17 and 15 cm. respectively. Compare the specific gravities of the two.

(8) If, in last question, the heavier of the two liquids is water, find the pressure of the air in the stem when the height of the barometer is 75 cm.

(9) A room has a volume of 150 cubic yards. The barometer rises from 28 to 30 inches. Find how many cubic yards of air in the room at the higher pressure have entered during the rise.

(10) In a common air-pump, the volume of the receiver is 450 cubic inches, and that of the barrel is 50 cubic inches. Find in what ratio the density is reduced by one stroke and by two strokes.

(11) The barrel of an air-pump is $\frac{1}{2}$ a cubic foot in volume. Find the pressure in a receiver 8 cubic feet in volume, after 5 strokes of the pump, in terms of its initial pressure.

(12) A barometer which has a little air in it reads 29·6 inches, the top of the tube being 6 inches above the top of the mercury, when a standard barometer reads 30. What is the true reading when the faulty one marks 29?

(13) Show that the apparent weight of a diving-bell increases as

it is sunk to greater depth in water, provided no fresh air is pumped into it. At what depth approximately will the sustaining force or buoyancy of the water on it be half what it is when just submerged, assuming its walls thin and about 3 feet high?

(14) A diving-bell 10 feet high is sunk in water so that its top is 287 feet below the surface. Find the height to which the water will rise inside the bell, assuming the pressure of the atmosphere equal to that of a column of water 32 feet high.

(15) A diving-bell having a capacity of 125 cubic feet is sunk in salt water to a depth of 100 feet. If the specific gravity of salt water be 1·02, and the height of the water barometer be 34 feet, find the total quantity of air at atmospheric pressure required to fill the bell.

(16) A mass of gas kept at constant temperature has a volume of 10 cubic feet at a pressure of 20 inches of mercury. Find the pressure which will reduce the volume to 3 cubic feet.

(17) A pump cylinder 1 square inch in section and 6 inches long is used to condense atmospheric air into a reservoir of 1 cubic foot capacity, which is kept cool. What is the pressure after 100 strokes?

(18) A barometer tube contains air above the mercury column. On a certain day the mercury stands at 25 inches when the space above it is 6 inches long, and at 24 inches when the space is made only 5 inches long by letting the tube lower down into its cistern. Find the true atmospheric pressure.

(19) A cylindrical vessel, closed at one end only, is 20 centimetres tall, and its open end is immersed in mercury until the interior level is 5 centimetres below that of the general level of the liquid outside. The barometric height being 75 centimetres, calculate how far the mercury has risen into the vessel, or how deep the lip of the vessel has been submerged.

MISCELLANEOUS EXERCISES.

SET I.

1. A bullet is fired vertically upwards with a velocity of 1600 feet per second. Find how high it will rise, and how soon it will hit the ground.

2. If you throw 2 balls up with a velocity of 192 feet per second, one two seconds after the other, when and where will they meet?

3. The intensity of gravity on Jupiter is 2·6 times as much as on the earth. How long would a body take to fall on Jupiter from a height of 167 feet?

4. A man in a stationary balloon throws a ball up with a velocity of 96 feet per second. Where is the ball in 10 seconds, and what velocity has it?

5. A man on a cliff 300 feet high throws a stone down to the ground in 3 seconds. With what velocity did he throw it?

6. A balloon is going up at the rate of 80 feet per second, and when at a height of 800 feet a halfpenny is dropped over its edge. What does the halfpenny do? When and with what velocity does it hit the ground?

7. A cannon is fired horizontally at a height of 10 feet above a lake. How soon does the ball hit the water?

8. One stone is dropped from the top of a cliff 400 feet high, and at the same moment another is thrown up from the bottom with a velocity sufficient to carry it to the top of the cliff. When and where do they meet?

9. A weight of 6 lb. is attached to one end of a string and 10 lb. to the other, and the string is hung over a freely movable pulley. Find the tension in the string; and how long 9 feet of the string take to pass over the pulley.

10. A knife is dropped from the middle of the ceiling of a railway-carriage going 50 miles an hour. How does it fall?

11. A train is going 50 miles an hour; a man, leaning out of window, throws a ball up vertically at 32 feet a second. What

becomes of it, and how long will it take before it comes back to his hand?

12. An iron cage descends a mine. The tension in the rope equals the weight of 200 lb.; when at rest it was 225 lb. Find the time of descending 100 feet from rest.

13. Find the tension in a rope which draws a carriage of 8 tons up a smooth incline of 1 in 5, and causes an increase of velocity of 3 feet per second per second. If on the same incline the rope breaks when the carriage has a velocity of 48 feet a second, how far will it continue to move up the incline?

SET II.

1. Six forces act on a point making angles of 60° with each other. Their magnitudes are 4, 6, 5, 1, 10, 7. Find, by drawing, the magnitude and direction of the resultant.

2. Three forces, 10, 10, 36, act on a point at angles of 120°. Find resultant.

3. A hundredweight is hung to 2 hooks in the ceiling by 2 cords, one 3 times as long as the other, and the same length as the distance between the hooks. Find by construction the tension in each.

4. A weight of 42 lb. is balanced at a height of 6 feet above the ground on 2 inclined rods meeting in a point under the weight. One rod supports 36 lb., the other 20 lb. Find the length of each rod by a construction.

5. Two boys sitting at the ends of a plank 10 feet long see-saw over a log. The plank weighs 70 lb. The log sustains 2 cwt., and one boy is 4 feet off it. What are the boys' weights?

6. Forces of 5, -3, 4, -2, 6, are arranged along a rod at equal distances (2 inches). Find resultant.

7. A uniform rod weighing 4 lb. has 12 lb. at one end and 18 at the other. The centre of gravity of the whole is 9 inches from the middle. What is the length of the rod?

8. Two men carry a block of iron weighing 176 lb. suspended from a pole 14 feet long. Each man is 1 foot 6 inches from his end of the pole. Where must the block hang in order that one man may bear $\frac{3}{4}$ of the weight borne by the other?

SET III.

1. State the characteristic difference between solids and fluids in relation to the transmission of pressure, and explain clearly

what is meant by the equal transmission of pressure by fluids in all directions.

2. Show what the pressure exerted by a liquid on any part of the surface of the containing vessel depends upon, and explain how to calculate the amount of this pressure when the necessary data are given.

3. Describe and explain an experiment proving that the pressure on the base of a vessel may be greater or less than the weight of liquid in the vessel.

4. Prove that the resultant pressure on a submerged body acts vertically upwards, and is equal to the weight of a quantity of fluid equal in bulk to the body.

5. Describe experiments proving that the air has weight, and show how the weight of a given volume of air can be approximately ascertained.

6. Explain the construction and action of the barometer, and show how to ascertain the pressure per unit of surface exerted by the air.

7. In a barometer which contains a little air in the space above the mercury, this space amounts to 20 c. c. when the mercury in the tube is 70 centimetres above the mercury in the cistern; on lowering the tube, so that the mercury in the tube is only 67 centimetres above that outside, the space above the mercury measures 12·5 c. c. Find true barometric height.

SET IV.

1. Define the terms 'force,' 'work,' 'power,' 'energy,' 'momentum.'

2. What experiments or observations can you adduce to prove that the weight of a body is proportional to its inertia?

3. A light frictionless pulley with a string over it has 17 ounces hanging on one end of the string and 15 on the other. Calculate the tension in the string, and the acceleration of either mass.

4. Discuss the direct impact of two small spheres on one another in the light of Newton's law of motion, showing what happens to their separate momenta and energies—(*a*) when inelastic; (*b*) when elastic.

5. Calculate the position of the centre of gravity of a light square frame, six inches in the side, with weights at its four corners, proportioned to 5, 2, 7, 4.

6. Explain the common air-pump, and show how to calculate

the pressure of the residual air after a specified number of strokes.

7. A pressure is often specified as equal to 70 centimetres of mercury. Express this in absolute measure; for example, in dynes per square centimetre.

8. How can the specific gravity of sand be practically determined? Illustrate by an example.

9. A ladder, weighing half a hundredweight and 30 feet long, rests against a smooth wall, with its foot 15 feet from the bottom of the wall. Find the pressure on the wall and ground, taking the centre of gravity of the ladder as one-third of its length up.

10. A stone is thrown up with a velocity of 192 feet a second. Find how high it ascends, and how long it takes before returning to the hand. Find also its position three seconds after throwing.

11. A body, moving with uniform acceleration, describes 180 feet in the fifth second of its motion. Find its acceleration and the distance travelled in the five seconds.

12. A tricycle, weighing 80 lb., moving along a level road at the rate of 12 miles an hour, is stopped by the friction in the space of 60 yards. What must the average resistance to its motion have been?

13. Of three blocks of wood one is pivoted on a point, another rests on an inclined plane, and the third floats in water. Discuss the conditions necessary for equilibrium and for stable equilibrium in each.

14. How would you determine the coefficient of friction between two given flat surfaces?

SET V.

1. Define Acceleration, Inertia, Force, and Work, showing how each is measured, and giving the most important standards or units of each in present use. What is meant by the statement that $g=981$?

2. It is found experimentally that *in vacuo* all bodies fall through a given distance in the same time. What consequences can be deduced from this fact? If the distance were doubled, how much more time would the fall require?

3. A falling body is observed to describe 100 feet in the last second of its motion. Find how far it must have fallen, and also the time taken. Consider $g=32$, and neglect the resistance of the air.

4. What is meant by *centripetal acceleration?* Find the force necessary to cause a planet of mass m to revolve in a circle of radius r in a time T.

5. How have the masses (a) of the Earth and (b) of the Sun been ascertained?

6. Define moment of inertia. Find the time a solid cylinder will take to roll down an inclined plane 20 feet long, inclined at 30° to the horizon; the moment of inertia of the cylinder being $\frac{1}{2}mr^2$.

7. Explain clearly what is meant by the centre of oscillation of a swinging rigid body; and determine its position in the case of a uniform rod swinging about one end, its moment of inertia being $\frac{1}{3}mr^2$.

How has the intensity of gravity been accurately measured?

8. Three weightless rods are jointed together, the two free ends are pivoted to firm supports, and the middle rod is loaded at any point. Sketch the position of equilibrium which the system will take up, and show how to determine by construction the stress in each of the unloaded rods.

9. A weight rests on an inclined plane of given roughness. Find by construction the least force which will suffice to pull the weight up the plane, showing the angle at which it must act. Also find how much the plane must be tilted in order that the weight may slide down.

10. A rectangular block weighing 20 lb. with a square base 8 inches in the side, is set up on a level table; and it is found that a horizontal force of 5 lb. weight, if applied below a certain point, is just able to make it slide, while if it is applied above that point the block topples over. Find the position of this critical point, and also the coefficient of friction between the block and the table.

11. Define density and specific gravity.

A piece of iron weighing 84 grammes is put into a beaker, which is then filled with water up to a certain mark above the level of the iron, and the whole is found to weigh 128 grammes. The iron is then turned out, and water poured in till the beaker is again full up to the same mark; it now weighs 56 grammes. Find the specific gravity of the piece of iron, and its weight when under water.

12. Describe an accurate form of barometer; and give some of the contrivances which have been employed to make barometers more sensitive to slight changes of pressure.

13. A balance is arranged under water, and a mass of iron ore in one of its pans is counterpoised by 3 kilogrammes of lead in the other. What is the mass of the iron ore, its specific gravity being 7, while that of lead is 11?

14. A ladder standing on rough horizontal ground rests against a rough vertical wall. Find its position when just not slipping down.

15. A solid regular triangular pyramid is drawn along a rough table by a horizontal force. What is the greatest height at which the force can be applied without upsetting the pyramid?

16. A weight is swung round and round in a vertical circle by a rope of given length. Determine the conditions that the rope may keep tense.

SET VI.

1. Describe fully a method of comparing the specific gravities, (*a*) of a solid and liquid, (*b*) of two liquids.

2. Explain clearly why a diving-bell which is not supplied with additional air appears to get heavier as it descends in water; and show how its depth might be ascertained, either by reading a barometer inside the bell, or by noticing the height to which the sea-water had risen into its interior.

3. A gun is fired horizontally, at a height of 169 feet above a lake, with an initial velocity of 1000 feet a second. Find how soon, and how far away, the ball will first strike the lake, neglecting the resistance of the air, and taking the acceleration produced by gravity as 32 feet-per-second per second.

4. What is the principle of *virtual work* (or *virtual velocities*)? Illustrate it by applying it to find the mechanical advantage of any system of pulleys when the weights of the pulleys are not neglected.

5. A mass of 3 lb., hanging vertically, drags a mass of 17 lb. along a perfectly smooth level table by means of a string over the edge. Find the acceleration, and the distance travelled in five seconds.

6. Find the tension in a flexible rope which is passed round a single movable pulley supporting 20 lb., while to the free end of the rope 12 lb. is hung; and find the acceleration upwards of the 20-lb. weight (neglecting the mass of the pulley and of the rope).

Show that this may be done either by direct application of Newton's Second Law, or by a work-and-energy method.

7. State Newton's Third Law. If two spheres of given masses and coefficients of restitution impinge directly on each other with known velocities, show how to find the velocities after the impact.

Consider specially the case when the masses are equal.

8. A ball let fall on to a stone slab from a height of 16 feet, bounces the first time to a height of 9 feet. What is the coefficient of restitution, neglecting the resistance of the air? and how high will the ball bounce next time?

Find also the total distance it will travel before coming to rest.

9. Given the moment of inertia of a body about an axis through its centre of gravity, determine it about any other parallel axis.

10. A uniform rod of given length is swung as a pendulum about a given point in it. Find the length of the equivalent simple pendulum, and find for what point of suspension the time of swing will be a minimum. [The moment of inertia of a rod about its centre is $\frac{1}{12}ml^2$.]

11. A conical pendulum or governor ball (considered as a particle) is spinning round a vertical axis 20 times a second. Find its distance in inches or centimetres below a horizontal plane through its hinge.

12. How has the value of g been determined accurately? Explain its variation with latitude.

13. Define the unit of force. How has the force of attraction between two pound masses one foot apart been determined?

Show how a knowledge of this would enable us to express the mass of the earth in terms of its bulk, and also would tell us the mass of the sun in terms of its distance.

14. What is the law of acceleration to which a body is subject if it is dropped into a deep hole in the earth? How long would it take to reach the centre, if the density of the earth is 5·7. Show that this time is independent of the size of the earth.

15. A body slides down a plane inclined at a given angle to the horizon. Determine its acceleration, and the time taken to slide down, supposing the coefficient of friction constant.

Determine also the least force necessary to support the body, and the direction in which it must act.

16. A uniform narrow beam is pivoted at one end at a given height above a pond, and the other end rests in the water. Determine its position of equilibrium, the specific gravity of the wood being known.

Discuss its change of position if the level of the pond is gradually rising towards, and ultimately above, the pivot.

SET VII.

1. A ball thrown up is caught again in 7 seconds. How high did it go, and with what speed was it thrown?

2. A cannon-ball is fired horizontally with a velocity of 1000 feet a second across a lake from a tower 100 feet high. When and where does it strike the water?

3. A 3-cwt. cage is being lowered down a coal-pit with a downward acceleration of 5 foot-second units. Find the tension in its rope.

4. Equal weights of 21 lb. each are slung by a string over a perfectly smooth bar, and 3 lb. is added to one side. Find the distance moved and the velocity acquired in one second from rest, and find also the tension in the string.

5. A gun weighing 12 lb. fires a bullet of an ounce and a half. If the initial recoil velocity is 13 feet a second, what is the speed of the bullet?

6. The above bullet at the end of a string 5 feet long is whirled round and round till the string breaks. If the string had been able to support half a hundredweight, what was the rate of whirling of the bullet?

7. A number of particles slide down different chords of a vertical circle, all of them meeting at the lowest point of the circle. Compare their speeds and times of descent.

8. Show that if the planets described circles round the sun, under the action of gravitation, the distances cubed would be proportional to their periods squared. How can the mass of the sun be thus calculated, in terms of the mass of the earth?

9. What is the value of g where a simple pendulum 7 feet long makes a complete swing in 3 seconds?

10. Find the range, the time of flight, and the greatest elevation of a projectile fired at 45° with a speed of 1000 feet a second.

11. What is the necessary difference of tensions in a driving-belt 30 inches wide, which is running 4200 feet a minute and transmitting 300 horse-power?

SPECIMEN EXAMINATION PAPERS.

SOUTH KENSINGTON EXAMINATION.—SOLIDS.

1. One of two forces, which act at a point, is represented numerically by 7; the resultant is 14 and makes an angle of 30° with the force of 7; find graphically the magnitude and line of action of the second force. Also calculate the magnitude to two places of decimals, and measure the angle between the two forces as accurately as you can.

2. Define the moment of a force with respect to a point.

The moments of two forces are taken with respect to a point; explain under what circumstances the moments will be of different signs.

Draw a square ABCD, and let a force of 12 units act from C to D, and another force of 23 units from C to B; find the moment of their resultant with respect to the point A.

3. Explain what is meant by the tension of a thread.

Two men pull at opposite ends of a rope, and each pulls with a force of 50 lb. weight; what is the tension of the rope?

A body weighing 10 lb. hangs at one end of a thread, the other end of which is fastened to a hook in the ceiling; what is the tension of the thread, and what are the forces that produce it?

4. A circular lamina of radius b centimetres weighs W lb.; find the common centre of gravity of the lamina and of a weight of W lb. distant a cm. from the perimeter and external to it.

When is the centre of gravity (a) inside the perimeter, (b) on the perimeter, (c) outside the perimeter?

5. Let a horizontal line AC represent a rod 12 ft. long, resting on two fixed points A and B, 10 ft. apart. Each foot of the length of the rod weighs 12 oz.; a weight of 16 lb. is hung from C. Show that the rod will stay at rest, and find the pressure at each of the points of support.

6. Explain the principle of the lever.

Describe the common balance, and show that the weights in the scale-pans will not be equal unless the beam is horizontal.

7. A piece of wire is hung up by one end, and at the other carries a weight. State the law which regulates the amount by

which the wire is stretched, and what limit there is to the extent of the elongation.

The wire is 12 ft. long, and its cross-section is 1-120th of a square inch; it is found to be stretched one-fifth of an inch by a weight of 300 lb.; find the modulus of elasticity.

8. When the acceleration of the velocity of a body moving in a straight line is constant, how is the acceleration measured?

In what units is acceleration commonly measured?

A body moves in a straight line, and at a certain instant its velocity is 10 ft. a second; at the end of $1\frac{3}{4}$ seconds its velocity is 31 ft. a second; at the end of $3\frac{1}{2}$ seconds its velocity is 52 ft. a second. Show that this is consistent with a constant acceleration, and find what that acceleration is in feet and seconds.

9. A train has a speed of 30 miles an hour at a certain station, and moves with a constant acceleration of $\frac{1}{4}$ foot per second per second till it passes a station a mile distant; what time does it take between the stations, and what is its average velocity?

10. Define a poundal and a foot-poundal.

The mass of a body is 15 lb. and its velocity is 20 ft. a second; find its kinetic energy in foot-poundals. Find also the number of poundals in the force that would bring it to rest in 1-10th of a second. What would the force be in pounds-weight?

11. Describe Atwood's machine.

Weights of P and Q lb. are connected by a string passing over a smooth fixed pulley, whose plane is vertical; determine the ratio of P to Q, so that the acceleration of the system may be one-tenth of that due to gravity.

12. A small but heavy body is suspended by means of a long, inextensible thread, and is able to swing to and fro in short arcs. How are the times of successive swings connected?

Find the length of the thread when a complete oscillation takes half a second, g being 32·2. $(\pi^2 = 9{\cdot}87.)$

SOUTH KENSINGTON EXAMINATION.—FLUIDS.

1. A body moves in a straight line; explain how its velocity is measured at any instant—(*a*) when the velocity is constant, (*b*) when it is variable.

A body falls freely at a place where $g = 32{\cdot}2$; find its velocity at the end of 5 seconds; also find its velocity when it has fallen through 144·9 ft.

2. Enumerate forms of energy produced in material bodies when work is done upon them.

If a one-ounce rifle-bullet be propelled with a velocity of 1500 ft. per second, what is its kinetic energy?

What work must be done upon it to produce this energy?

3. State broadly the differences between solids, liquids, and gases. How do they behave in general when attempts are made to compress them?

Make remarks on the statement: 'The pressure of a liquid at rest is always perpendicular to every surface on which it acts.'

4. There is a cylindrical vessel 2 ft. high, closed at both ends and standing upright. Suppose that a pipe is let into the side of the vessel and extends upward to a vertical height of 12 ft. above the top of the vessel. Now suppose that water is poured down the pipe until the vessel is full, and that the water stands in the pipe at a height of 10 ft. above the top of the vessel. If you were considering the pressure of the water on the inside of the vessel, where would you take the surface of the water to be? Bearing your answer in mind, find the pressure on each square foot of the top and of the bottom of the vessel.

Assuming that the height of 10 ft. is not changed, explain whether the results would or would not be different if the diameter of the tube were doubled, and if it came into the vessel above or below the point at which it actually enters.

5. A vertical side of a rectangular cistern is 4 ft. broad by 5 ft. deep; the cistern is $\frac{4}{5}$ths full; what is the magnitude of the resultant pressure of the water on the side, and where and how does it act?

6. State the conditions of equilibrium of a floating body.

A solid body floating in water has $\frac{1}{8}$th of its volume above the surface. What fraction of its volume will project if it float in a liquid of specific gravity 1·2?

7. A cylinder, whose specific gravity is 0·6, is fastened to the bottom of a vessel by a thread attached to a point in the circumference of its base; its radius is 2 in. and height 6 in. If water is poured into the vessel, and the body comes to rest entirely under water, show, in a carefully drawn diagram, the position in which it comes to rest, and find the tension of the thread.

8. Explain how to find the specific gravity of a substance lighter than water, by means of the balance.

A piece of wood weighs 15 oz. and its specific gravity is 0·8;

when it is attached to a sinker, the compound body weighs 16 oz. in water; find the weight of the sinker in water.

9. Give a rule for converting the reading of a Fahrenheit thermometer into that of a Centigrade.

What reading Centigrade corresponds to 98·4 F.?

10. Given that, at the standard pressure and temperature, the weight of a cubic inch of dry air is 0·309 grains, and that a cubic foot of water weighs 62·4 lb., show that the density of water is very nearly 818 times that of the dry air.

A substance of small specific gravity is weighed by brass weights; explain whether the true weight of the substance is greater or less than the apparent weight.

11. Explain what is meant by the water barometer.

Describe a siphon, explain its action, and state what are the limits within which it will act.

A siphon is working, when a small hole is made at the top of the bend; explain what follows.

12. If 248 lb. of air are contained in a room whose dimensions are 18, 15, and 12 ft., the temperature being 60° F. and the pressure that due to 30 in. of mercury, find the weight of one cubic foot of air at 37° F. under the pressure due to 29·7 in. of mercury.

South Kensington Examination.—Solids.

1. Two forces act at a point along given straight lines; explain how to find the force that will balance them.

Let AB and AC be the lines, and let the angle BAC be 51°; let a force of 27 units act from A to B, and one of 41 units from A to C; find by construction the force that will balance them, and, if it act from D to A, find (in degrees) the angle BAD.

2. Two men carry a block of metal suspended from a pole 7 feet long, each man being 6 inches from his end of the pole; find the point of suspension when one man bears $\frac{5}{7}$ths of the weight borne by the other.

3. What is the 'moment' of a force about a point? An hexagonal lamina ABCDEF is fixed at the centre and capable of rotation about an axis perpendicular to its plane. A force of 1 lb. weight is applied at A perpendicular to AB and in the plane of the lamina; find the magnitude of the force along CD which will prevent rotation, and show the directions of the two forces by arrow-heads in a diagram.

4. A body is hung up by a piece of string. What do we know about the position of its centre of gravity?

A triangular board, ABC, is hung up by a thread fastened to A; show, in a carefully drawn diagram, the position in which it will come to rest.

If the board weighs 1 lb. and a weight of 1 lb. is fastened to B, show, in a second diagram, the position in which it will now come to rest.

5. A, B, C are fixed smooth points, such that ABC is an equilateral triangle, with BC horizontal and above A. A fine thread is fastened to A, passes over B and C, and carries a weight of 10 lb.; find the pressures on B and C, produced by the weight.

6. Define a foot-pound of work. State what is meant by the power of an agent.

There are two agents A and B; A can lift 25 lb. through 6 feet in a tenth of a second; B can lift 10 tons through 15 feet in 70 seconds; find the ratio of A's power to B's.

7. How are two velocities compounded?

A boat is rowed at right angles to the stream of a straight river one-third as fast again as the river flows. If the distance rowed from bank to bank be five miles, find the breadth of the river and the distance made in the direction of the stream.

8. A body moves from rest in a straight line, and its velocity is uniformly accelerated; show how to represent graphically its velocity at end of a given time, and the total distance described.

A body is moving at the rate of 80 feet a second; find the height to which that velocity is due.

9. In the motion of a body in a straight line, define constant velocity and constant acceleration, and show that the former is a particular case of the latter.

In the same system of units the constant velocity of one body and the constant acceleration of another body are denoted by the same number; if the second body start from rest, show that in two seconds both bodies will have passed over the same distance.

10. Distinguish between potential energy and kinetic energy.

How many units of work are necessary to raise 4·7 ounces 5·3 feet high?

11. Explain how it is that a body moving with uniform velocity in a smooth horizontal circular groove can have at every instant an acceleration, constant in magnitude, directed towards the centre of the circle.

How is this acceleration produced? What is the magnitude

of the force which produces the acceleration? If the body weigh 2 oz., the diameter of the circle be a yard, and the uniform velocity be 5 feet per second, express the force in poundals and in pounds-weight.

12. State what is meant when it is said that, at a certain place, the acceleration due to gravity is 32·2.

Find the number of beats made at that place in one minute by a pendulum 20·125 feet long.

SOUTH KENSINGTON EXAMINATION.—FLUIDS.

1. Define the British absolute units of force, and show how force is measured in any particular case of a body's motion.

A body weighing 2 lb. at rest is acted upon by a force of 7 poundals for 3 seconds; what velocity does it acquire?

2. A body whose mass is 264 lb. moves at the rate of 1000 ft. per second; find its kinetic energy (a) in foot-poundals, (b) in foot-pounds. ($g=32$.)

If that energy were imparted to a machine, and were able to make it work uniformly for 5 minutes, with what horse-power would the machine be working during those five minutes?

3. Define the centre of gravity of a body. State where the centre of gravity is situated in the case of any two bodies.

A rope unwinds from a drum, and the hanging end descends vertically at the rate of 10 ft. a second; with what velocity does the centre of gravity of the hanging part of the rope descend?

4. Explain how the pressure of a fluid is measured.

If you had a cylinder standing upright and containing water 2 ft. deep, explain how you would estimate its pressure at any point of the base. How would your statement be modified if you considered a point on the side of the vessel 1 ft. below the surface of the water? (7 cub. in. of water weigh 4 oz.)

5. What is the resultant pressure on a body wholly immersed in water?

State, with reasons, if this depends (i.) upon the weight of the body; (ii.) the size or shape of the body; (iii.) the depth to which it is immersed.

Find the force with which a sphere 1 foot in diameter is urged upward if it be totally immersed in water.

6. A cube whose edge is 6 inches is placed in water with a face horizontal and its centre of mass at a depth of 3 feet. Find the pressure in lb. on each face of the cube.

7. There are two liquids, which do not contract when mixed, and their specific gravities are 0·6 and 0·75. If equal volumes of them are mixed, what is the sp. gr. of the mixture? If equal weights of them are mixed, what is the sp. gr. of the mixture?

8. Describe a 'specific gravity bottle.'

Such a bottle when empty weighs 20 grammes; when full of mercury it weighs 141 gr., and when full of another liquid it weighs 73 gr.; compare the specific gravities of the two liquids.

9. State Boyle's law, and describe the experimental verification in the case of air, when the pressure is increased.

The barometer stands at 30·1 in.; there is no change of temperature, but the barometer falls to 29·025 in. What will now be the volume of the quantity of air which at first had a volume of one cubic yard?

10. A cylindrical tube closed at one end is held in a vertical position and immersed mouth downwards in water; what is the depth of the middle of the tube when the water has risen half-way up the tube, the atmospheric pressure being 15·1 lb. weight per square inch?

11. Draw a diagram of a compressed-air manometer, and explain the principle involved in the construction.

12. Describe the hydrometer of variable immersion.

The hydrometer when floating in a liquid (A) has more of the stem above the surface than it has when floating in another liquid (B). Explain which of the two liquids is the denser.

One inch of the stem of a hydrometer is one-fiftieth part of its whole volume. When it is placed in the liquid A, two inches of the stem are above the surface; when it is placed in B, one inch and a half are above the surface; compare the specific gravities of A and B.

Royal University of Ireland Matriculation Examination.

1. Explain the terms 'poundal,' 'dyne,' 'erg.'

2. State the laws of friction, and describe any method of determining by experiment the coefficient of friction.

3. Masses of 3, 4, and 5 lb. respectively are equally spaced along a straight line; find their centre of gravity.

4. A body is dropped vertically; what distance does it pass over during the third second?

5. Define 'momentum.' Compare the momentum of a mass of

7 lb. having a velocity of 1000 feet per second with that of a mass of one ton moving at the rate of 2 miles per hour.

6. Define the pressure at a point in a fluid.

7. A body floats with a cubic foot of its volume under water. What volume would be immersed in a liquid of sp. gr. 1·2?

8. A balloon is not completely filled with coal-gas, so that the skin is slack. Why, as it rises, does it fill?

9. Describe some method of finding the sp. gr. of a liquid.

10. Describe the hydraulic press, and calculate its mechanical advantage.

Royal University of Ireland Matriculation Examination (Honours).

1. How would you examine experimentally the sensitiveness of a balance? On what points in its construction does the sensitiveness depend? Give proofs of your statements.

2. A uniform triangular plate is divided by a line drawn parallel to its base. Find the position of this line if the distance between the centres of gravity of the two parts is equal to half the median.

3. Two moving bodies, A and B, are brought to rest by the application of the same retarding force to each. If A moves p times as long, and q times as far, as B before coming to rest, find the ratios of the masses, and of the speeds of the bodies.

4. State the laws which govern the direct impact of moving bodies, and explain how they could be tested experimentally.

5. In the arrangement of one movable and one fixed pulley, in which the theoretical advantage is two, find the accelerations of the parts, when a one-pound weight is hung on the free end of the string, and the movable pulley and its attached weight have a mass of three pounds.

6. Two particles are projected vertically at the same instant and from the same point, their velocities being any whatever. Show that their relative velocity remains unchanged until one of them strikes the ground.

7. Prove that the free surface of a liquid at rest under gravity is horizontal.

8. Calculate the thrust on a triangular lamina which is held in a liquid with its plane vertical and base horizontal, three-fourths of the area being immersed.

9. A piece of iron weighs 500 grammes. It is made into a

hollow spherical shell. What is the outside radius of the shell if (a) it just floats, (b) it floats half immersed?

10. A body of specific gravity s floats in water with a certain water-line; prove that if the body were inverted, and of specific gravity $1-s$, it would float with the same water-line.

11. Describe some form of double-cylinder force-pump.

12. A cylindrical tube three feet long and one square inch in sectional area is used as a barometer, the mercury standing at 30 inches. Find the effect of introducing into the Torricellian vacuum a volume of air which, at atmospheric pressure, measures five cubic inches.

UNIVERSITY OF LONDON MATRICULATION EXAMINATION.

1. A body falls freely from rest and reaches the ground with a velocity of 40 feet per second. Find the velocity with which it would reach the ground if, for 9 feet after passing the mid-point of its descent, the body were subject to no acceleration.

Would it be possible to reproduce approximately, in an experiment, conditions similar to those described above?

2. Explain what is meant by *relative velocity*.

Two small marbles A and B are moving in clockwise direction in concentric circular grooves of 2 and 3 inches radius respectively. The velocity of A in its groove is 2 inches per second, and that of B is 9 inches per second. At a given instant the marbles are 1 inch apart; what time will elapse before the distance between them is 5 inches?

3. A shot weighing 10 lb. is discharged from a gun weighing 10 tons. The shot escapes from the gun, with a velocity of 2000 feet per second, $\frac{1}{4}$ths of a second after the powder is fired. Calculate the velocity of the gun at the instant the shot leaves it, assuming that its motion is unimpeded.

What average force would it be necessary to exert upon the gun to keep it from changing its position during discharge?

4. A train is moving at a uniform speed of 40 miles per hour along a level track, and the pull exerted upon it by the engine is equal to the weight of two tons. How much work, in foot-lb., is expended by the engine per minute against frictional resistance to the motion of the train?

What further knowledge would be required in order to calculate the frictional resistance at any instant if the speed of the train were not uniform?

5. Explain the meaning of the terms *stable*, *unstable*, and *neutral equilibrium*.

A uniform triangular plate weighing 1 lb. has an extra weight of 1 lb. attached to it at one of its angular points. It is free to rotate about a horizontal axis at right angles to its surface and passing through the point of intersection of its median lines. Show in which positions the plate will be in stable and unstable equilibrium respectively. Has it any position of neutral equilibrium?

6. What is meant by the *mechanical advantage* of a machine?

Show how to calculate the mechanical advantage of a differential wheel and axle.

What practical advantage has such a wheel and axle over a simple wheel and axle of the same mechanical advantage?

7. Mercury is poured into a U tube which has a section of one square inch, and which is standing with its limbs vertical, so as to more than occupy the bend; 20 cubic inches of water are then poured into one limb. Through what height will the mercury rise in the other? (Specific gravity of mercury = 13·6.)

Explain how such an arrangement might be used to determine the specific gravity of a liquid. Is it necessary for this purpose that the two limbs should have the same sectional area?

8. Describe and explain the action of an air-pump in which exhaustion is effected by a piston.

Why should the valves of such an air-pump be made of a very light material?

Why is an air-pump with two barrels easier to work than one with a single barrel?

UNIVERSITY OF LONDON MATRICULATION EXAMINATION.

1. A heavy weight falls freely, being guided in its descent by a smooth vertical rod down which it slides. A pencil attached to the weight leaves a trace upon a sheet of paper rolled round a vertical cylinder which is made to rotate upon its axis with uniform speed. Show upon a diagram the trace of the pencil point, supposing the paper to be unrolled after the experiment, and show how from the trace you could prove that the weight had fallen with constant acceleration.

2. The diameter of a bicycle wheel is 30 inches, and it is supposed to be so geared that the wheel revolves twice for every revolution of the treadle. Determine the actual velocity of the treadle in magnitude and direction at each quarter of a revolu-

tion, starting from the lowest point, supposing the crank to be $7\frac{1}{2}$ inches long, and the bicycle to be moving forward on a horizontal track with a speed of 10 yards per second.

3. A stone is whirled round in a vertical circle at the end of a string 21 inches long with a speed that may be regarded as uniform, and makes a revolution every second. The centre of the circle is 4 feet above the ground. Determine the subsequent history of the stone if the string were to break (i.) at the instant that the stone was at the lowest point of its circular path, (ii.) a quarter of a revolution later.

4. A train of mass 200 tons travels with constant speed of 60 miles an hour in a northerly direction. It suddenly comes to a curve, on running over which for 10 seconds its motion has become deflected towards the east so that it makes an angle of 30° with north. What eastward velocity has the train acquired by running round the curve, and what easterly momentum? What force must have been applied to it to give it this momentum, and how has the force been applied?

5. An equilateral triangular lamina is suspended by threads fastened to two of its angular points. The directions of the threads produced pass through the centre of mass of the lamina, and one of them is horizontal and has a tension of 1 lb. Find the tension of the other thread, and the weight of the lamina.

6. A table consists of a uniform circular board weighing 9 lb., supported by three vertical legs fixed at equal distances round the circumference. A weight of 12 lb. placed on the edge of the table, midway between two legs, is just sufficient to cause the table to overturn. Find the weight of each leg.

7. Show that the resultant thrust on a solid immersed in a liquid acts vertically upwards through the centre of gravity of the displaced liquid.

A long, thin stick of uniform cross-section and density has a bullet, of weight equal to its own, attached to one end and floats, half immersed, in a vessel of water. Show that it is in neutral equilibrium.

8. Describe a siphon and explain its action.

How does the rate at which a vessel is emptied by a siphon depend upon (1) the lengths of the limbs, (2) their cross-section, (3) the density of the contained liquid, and (4) the pressure of the atmosphere?

ANSWERS TO EXAMPLES.

EXAMPLES I. *Pages* 10, 11.

(1) $8\frac{3}{11}$ seconds.
(2) $9\frac{1}{2}$ miles per hour.
(3) 134.
(4) $4\frac{1}{11}$.

Answers given in full on page 11.

(5) 603·2 ; $\frac{1}{4}$ inch.
(6) 75 feet.
(7) $10\frac{6}{27}$ seconds.
(8) 8·8 feet per second.

EXAMPLES II. *Pages* 14, 15.

(1) 20 ft./(sec.)^2 ; *i.e.* 20 feet per second per second.
(2) 8 ft./(sec.)^2.
(3) 52, 84, 116, 212 ft./sec.
(4) 4 miles an hour per minute, or 240 miles an hour per hour.
(5) $78{,}545\frac{5}{11}$.
(6) $1\frac{7}{15}$.
(7) -12 ft./(sec.)^2.
(8) In 4 seconds ; 96, 32, -32, -96 ft./sec.
(9) 16 seconds.
(10) $\frac{1}{4}$.
(11) 115,866.
(12) 6 inches/(sec.)^2 ; *i.e.* its velocity increases by 6 inches per second every second.
(13) $\frac{1}{4}$ mile/(min.)^2 = $\frac{11}{30}$ ft./(sec.)^2.
(14) $17\frac{3}{5}$ ft./(sec.)^2.

EXAMPLES III. *Pages* 20, 21.

(4) 24,960 miles.
(5) $1\frac{1}{2}$ radians-per-second per second.

(6) 3000 ft./sec. ; 5730 (nearly).
(7) 2·095.
(8) $19\frac{1}{16}$.
(9) 1·2 ; 11·46.
(10) 9·4 radians/sec. ; 16 miles/hour, or $23\frac{1}{2}$ ft./sec.
(11) 192 radians/sec. ; 48 feet/sec.
(12) 37·7 ft./sec. ; $50\frac{1}{4}$ radians/sec., or 480 revs. per minute.

Examples IV. *Pages* 29-31.

(1) In Question 1 the initial velocity is 0, the final is 600; therefore the average is 300; therefore in 30 seconds it will have gone 9000 feet. In Question 2 the average velocity is $\frac{1}{2}(50+102)=76$, so in $6\frac{1}{2}$ seconds it will go 494 feet; and so on. Or we may use the formula $s=v_0t+\frac{1}{2}at^2$. (3) 36, 104, 204, 696 feet respectively. (4) $\frac{5}{8}$ of a mile. (6) $\frac{2}{9}$ mile $=391\frac{1}{9}$ yards. (7) 304 feet. (8) The respective distances from the starting-point are $s_1=s_7=112$ feet; $s_3=s_5=240$ feet; $s_4=256$ feet. (9) 4096 feet.

(2) $52\frac{1}{14}$ ft./(sec.)^2.
(3) 1·96 ft./(sec.)^2.
(4) A and B, -2; C, $+2$; D, $-11·25$.
(5) A, 5 seconds; B and C, 10 seconds; D, $10\frac{3}{8}$ seconds. The diagram for C will look like fig. 18 upside down.
(6) It will continue moving for 26 seconds with an average velocity of 39 feet per second, and therefore will go 1014 feet.
(7) The acceleration must be 4, because the average velocity during the first second spoken of, and therefore the velocity in the middle of that second, is 16, while in the middle of the next second it is 20. The velocity at the beginning of the former of these seconds must therefore have been 14, and at the end of the latter second 22. To gain the velocity 14 with acceleration 4 required $3\frac{1}{2}$ seconds, which is therefore the time the point had been moving at the beginning of the first second spoken of. Starting with velocity 22, it would go 552 feet in the next 12 seconds; and its velocity would be 128 in 32 seconds from the original start, or $26\frac{1}{2}$ seconds from the time its velocity was 22.
(8) In another $11\frac{1}{4}$ yards, which will take $2\frac{1}{4}$ seconds.
(9) 33 miles; done in 36 minutes.

(10) Nearly 2 miles/hour per second, or 2·87 ft./sec. per second.
(11) 1 mile an hour per second; $8\frac{4}{5}$ miles an hour per second; 176 miles an hour per second, $=258$ ft./(sec.)^2.
(12) 400 feet; 176 feet.
(13) $210\frac{1}{4}$ feet; $3\frac{5}{8}$ seconds.
(14) 160 feet; 90 feet.
(15) See Ex. II. 8, and Ex. IV. 1.
(16) 25 feet; $2\frac{1}{2}$ seconds; 20 feet.
(17) 25 feet from the top, in $1\frac{1}{4}$ second.
(18) 100 feet; 80 feet per second.

EXAMPLES V. *Pages* 36, 37.

(1) It moves with a uniform velocity 50 in a straight line nearly NE. by N.
(2) 3 miles; $\sqrt{13}$ miles.

EXAMPLES VI. *Pages* 41, 42.

(1) Shortest distance, 352 yards; time, 4 min. 12 sec. One of the ships will have still ·16 miles to go before reaching the crossing-point, the other will have passed it by ·12 miles.
(2) In the first case, she will cross the middle of the road 5 yards in front of the vehicle, though it will afterwards pass within 1·6 yards of her (that is, $\frac{1}{2}\sqrt{10}$ yards). In the second case, by choosing a direction perpendicular to the line joining their initial position, she can just get across at 2·6 miles an hour without being run over.
(3) *Relative velocity*, 35 miles an hour; *minimum distance*, about 173 yards.
(4) 50·3 feet per second, its easterly component being $10(1+\sqrt{2})$, and its northerly component $10(3+\sqrt{2})$.
(5) 2 feet per second in a direction 30° west of south.
(6) $48\frac{1}{2}$ miles from its starting-point.
(7) 10·6 miles away from its correct line; $50\frac{1}{2}$ miles from its starting-point.

EXAMPLES VII. *Pages* 54, 55.

(1) 139 lb. (ft.)^2.
(2) 41830 lb. (ft.)^2.

(3) About 876,300 F.P.S. units.

(4) In lb. (ft.)^2 units: (*a*) $10\frac{5}{12}$; (*b*) $41\frac{2}{3}$; (*c*) 15; (*d*) $26\frac{2}{3}$; (*e*) $3\frac{3}{4}$; (*f*) $6\frac{2}{3}$.

(5) $\frac{1}{25}$ kgm. (metre)^2.

(6) The moment of inertia of the hollow sphere is $\frac{2}{3}$ that of the hollow cylinder; the moment of inertia of the solid sphere is $\frac{4}{5}$ that of the solid cylinder.

(7) 55 lb. (ft.)^2.

(8) $6\frac{1}{4}$ lb. (inch)^2.

(9) $2\frac{1}{12}$ lb. (inch)^2.

(13) Moment of inertia of a hollow sphere whose external radius is r, and internal r', is $\frac{2}{5}\rho(\frac{4}{3}\pi r^5 - \frac{4}{3}\pi r'^5)$, ρ being its density. Its mass $=\rho(\frac{4}{3}\pi r^3 - \frac{4}{3}\pi r'^3)$.

$\therefore$ its moment of inertia may be written $= \frac{2}{5} m \times \frac{r^5 - r'^5}{r^3 - r'^3}$

$= \frac{2}{5} m \frac{r^4 + r^3 r' + r^2 r'^2 + r r'^3 + r'^4}{r^2 + r r' + r'^2}$, which, when $r' = r$ (as is the case in the limit), reduces to $\frac{2}{5} m \times \frac{5r^4}{3r^2} = \frac{2}{3} m r^2$.

EXAMPLES VIII. *Pages* 65, 66.

(1) 9 units of acceleration; 450 units of length.

(2) 36 poundals, about equal to the weight of 1 lb. 2 oz.

(3) 2 ft./(sec.)^2; 70 units.

(4) 1 minute.

(5) 6250 yds., *i.e.* about $3\frac{1}{2}$ miles.

(6) 10 seconds.

(7) 39,240 dynes = $39\frac{1}{4}$ kilodynes = $\frac{1}{25}$ of the weight.

(8) 60 poundals, which is about the weight of 1 lb. 14 oz.

(9) 600 units.

(10) 75 feet.

(11) $37\frac{1}{2}$ lb. weight.

(12) About $1019\frac{1}{3}$ grammes weight, or nearly 2 per cent. more than the weight of a kilogramme; 27,810 dynes will support an ounce, and $996\frac{3}{4}$ million dynes will support a ton.

(13) 192 ft./sec.; between 870 and 880 miles.

(14) 14 centimetres per sec.

(15) 300 dynes.

(16) (*a*) $37\frac{1}{2}$ lb. weight; (*b*) $62\frac{1}{2}$ lb. weight; (*c*) 50 lb. weight.

(17) Its own weight.

EXAMPLES IX. *Pages* 71-73.

(1) $6\frac{1}{3}$ feet per second.
(2) 2 feet per second.
(3) $31\frac{1}{2}$ ft./sec. (assuming the clay to stick to the body).
(4) $\frac{7}{9}$.
(5) 7·2 and 9·7 ft./sec.
(6) 1600 ft./sec.
(7) 1 ft./sec. nearly.
(8) $11\frac{2}{3}$ and $21\frac{2}{3}$ ft./sec.
(9) The velocity of the larger body will be reduced to 2 ft./sec., while the smaller body will have a reversed velocity of 7 ft./sec.
(10) 7 lb.; reversed velocity of 5 ft./sec.
(11) $\frac{1}{7}$.
(12) 607 kilogrammetres, being over 95 per cent. of the initial energy.
(13) $\frac{3}{4}$; $5\frac{1}{16}$ ft.; $57\frac{1}{7}$ *ft.*
(14) Two cases may be drawn. In one, the direct component of the velocity of the heavier sphere, A, is changed from $+17·32$ to $-11·4$ ft./sec., while its transverse velocity remains 10 ft./sec., giving a resultant 15·17 ft./sec.; and the direct component of B's velocity is changed from -15 to $+20·9$, its unchanged component being $15\sqrt{3}$, and the resultant 33·34 ft./sec. In the other case, the direct component of A's velocity changes from $+10$ to -22, its transverse component being $10\sqrt{3}$, and the resultant 28; while the direct component of B's velocity changes from $-15\sqrt{3}$ to $+14$, its transverse component being 15, and the resultant 20·5.
(15) $4\frac{1}{2}$ ft./sec. (nearly).
(16) 500 poundals, *i.e.* about $15\frac{1}{2}$ lb. weight.
(17) 230 poundals, *i.e.* about 7 lb. weight.
(18) The kick felt is the jerk of maximum force at the relaxation of the accumulated pressure. The speed of the bullet may have to be mainly acquired in the last inch of the barrel instead of being gradually imparted over the whole 4 feet.

EXAMPLES X. *Pages* 82, 83.

(1) $27\pi^2$ poundals, equivalent to the weight of about 8·4 lb.

(2) (Taking π^2 as equal to 10) 5 feet.
(3) 6 inches.
(4) 80 ft./sec.
(5) $5\frac{1}{2}$ tons weight.
(6) 140 lb. (approx.).

EXAMPLES XI. *Pages* 96–98.

(1) 160 poundals, or $\frac{1}{4}$ of their usual weight.
(2) At a height equal to the earth's radius.
(3) In $15\frac{2}{3}$ seconds.
(4) 981.
(5) $44\frac{4}{9}$ grammes weight, or 43,600 dynes.
(6) 2 ft./(sec.)^2; 90 poundals, or 45 oz. weight.
(7) 6 inches-per-second per second.
(8) 224 feet.
(9) 4·4 seconds.
(10) 20 poundals per lb.
(11) 60 poundals; 2 ft./(sec.)^2.
(12) 2 ft./(sec.)^2; 510 poundals.
(13) $\frac{1}{10}g$; *i.e.* 3·2 ft./(sec.)^2, or 98 centimetres/(sec.)^2.
(14) 12 feet; 8 ft./sec.; $146\frac{2}{3}$ and $138\frac{2}{3}$ poundals.
(15) 4 lb.
(16) 12 feet; $117\frac{1}{3}$ poundals.
(17) 1 second. 46 ft./sec. *up*, or 14 ft./sec. *down*. Rising with a velocity of 48 ft./sec.
(18) The weight of 3 tons.
(19) The weight of $3\frac{9}{16}$ tons.
(20) The weight ascends at the same rate; and whatever the monkey does the weight does the same.

EXAMPLES XII. *Pages* 101, 102.

(1) $7\frac{1}{2}$ seconds.
(2) 1·875 mile.
(3) The tension in the string being greatest at its lowest point (because of gravity), the string is most likely to break there. The centrifugal force being equal to the weight of 100 lb., the velocity of the stone when the string breaks must be 40 feet per second. It will start forward hori-

zontally with this velocity and describe a portion of a parabola—striking the ground after the lapse of half a second 20 feet away.

(4) In 2½ seconds; 100 feet from where it would have dropped if the balloon had been stationary; with 80 ft./sec. vertical, and 40 ft./sec. horizontal velocity.

(5) 600 feet, 225 feet.

(6) 900 ft., 1800 ft.; 3 seconds.

(7) 10 seconds; 400 feet.

(8) $40\sqrt{6}$ ft./sec.; 45°.

(9) $40\sqrt{5}$ ft./sec.; 25 ft.; 2½ seconds.

(10) 40 ft./sec.

(11) 20½ miles; 15,625 feet, or nearly 3 miles.

(12) 100 ft./sec.

(13)

Angle.	Seconds.	Distance in feet.	Maximum elevation in feet.
0	2½	3000	100
30°	37½	39000	5625
45°	53	45000	11250
60°	65	39000	16875
−30°	⅛	173	100
Dropped	2½	0	100

EXAMPLES XIII. *Pages* 105, 106.

(1) 0·999785 of a second.

(2) 32·0763.

(3) 216.

(4) 261; 99·396 centimetres.

(5) 87¾ inches.

(6) 8531.

(7) $\frac{13}{340}$ of an inch.

(8) $\frac{g}{\pi^2}$; that is to say 39·14 inches, or 99·4 centimetres.

(9) 32·0763 poundals per lb.

(10) 32·128.

(11) 0·3009, 0·5015, and 0·4012 sec. respectively.
(12) $62\frac{1}{2}$.
(13) $\omega = 4$ radians per sec.; $v = 8\sqrt{3}$ ft./sec.; $t = \frac{1}{2}\pi$ seconds.

EXAMPLES XIV. *Pages* 109, 110.

(1) 24 inches.
(2) $6\sqrt{3}$ inches (*i.e.* about 10·4 inches) from the middle; 1·46 seconds.
(3) At a distance from the hinge $= \frac{2}{3}$ of the breadth of the door.
(4) 1284 ft./sec.
(5) $1\frac{1}{6}$ seconds; $13\frac{1}{3}$ inches.
(6) $2\frac{1}{40}$ seconds.
(7) At any point 10 inches from its centre. It will swing quickest about an axis distant 17·32 inches from the centre.
(8) *Length* of simple pendulum $= \frac{1}{2}h$; *Mass* $= \frac{2}{3}m$.
(9) *Length* same as in No. (8); *Mass* $= \frac{1}{3}m$.
(10) Time of complete swing $= 1·88$ sec.
(11) 3 lb.
(12) 150.
(13) $\frac{\pi}{4}\sqrt{6}$ seconds.
(14) 1280 ft./sec.

EXAMPLES XV. *Pages* 128–132.

(1) $16\sqrt{2}$ feet per second.
(2) 2400 foot-poundals, or about 75 foot-pounds.
(3) 4 ft./sec.
(4) (By equating the initial and final energies) 615·8 ft./sec.
(5) 689 feet per second.
(6) $\frac{54}{88}$th of the weight of the projectile.
(7) 400 poundals; $\frac{3}{16}$ second.
(8) $93\frac{3}{4}$ lbs. weight.
(9) $217\frac{1}{2}\,\pi^2$ foot-poundals, or about 68 foot-pounds.
(10) $28\frac{1}{4}$ million foot-poundals, or about 880 thousand foot-pounds, *i.e.* nearly 400 foot-tons.
(11) About 15,600 foot-tons.
(12) 2000 foot-poundals, or about 62 foot-pounds.

(13) 100,000 foot-pounds per minute, or about 3 horse-power (see No. 19).

(14) $13\frac{3}{4}$ tons weight (nearly).

(15) 155 foot-pounds; energy at lowest point, 45 foot-pounds,

$$\therefore 45-\frac{155x}{10}=20x \therefore x=\frac{90}{71}, \text{ i.e. it rises } 1\tfrac{19}{71} \text{ feet.}$$

(16) $171\frac{1}{2}$; 236.

(17) 6·7 tons weight.

(18) $\frac{7}{320}$ of the weight of the truck = the weight of $1\frac{5}{16}$ cwt.; 40 ft./sec.

(19) $98\frac{2}{11}$.

(20) $35\frac{7}{11}$ lb. weight.

(21) 314·16 foot-pounds; $141\frac{1}{2}$ British units.

(22) *Energy* = 1 foot-ton; *momentum* = 5666·8 British units.

(23) $\frac{1}{16}$ of a horse-power.

(24) $37\frac{1}{2}$ lb.

(25) 6400 feet.

(26) 600 (nearly).

(27) 460 (nearly).

(28) 0·17; 3·6 seconds; 36 ft.

(29) $322\frac{3}{4}$ tons weight.

(30) 264 foot-tons; (*a*) 5·6 lb. weight; (*b*) 7 lb. weight.

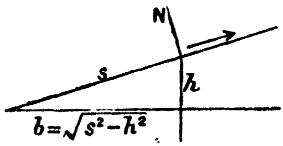
(31) 3640 foot-pounds; 2180 foot-pounds.

(32) The sliding body.

(33) Energy of roller $=mgh=\frac{1}{2}mv^2+\frac{1}{2}I\omega^2=\frac{1}{2}mv^2+\frac{1}{4}mr^2\omega^2$. [See list in Sect. 43.]
$=\frac{3}{4}mv^2$.

Energy of slider $=\frac{1}{2}mv^2$

$$=mgh-\left(mg\frac{b}{s}\right)\mu s$$
$$=mgh-mgb\mu.$$
$$\therefore mgb\mu=mgh-\frac{1}{2}mv^2$$
$$=\frac{1}{3}mgh.$$
$$\therefore \mu=\frac{1}{3}\cdot\frac{\text{height}}{\text{base}}=\frac{1}{15}.$$



$$\left(\text{The general solution is } \mu=\frac{I}{mr^2+I}\cdot\frac{\text{height}}{\text{base}}.\right)$$

(34) Energy $=mgh=\frac{1}{2}mv^2+\frac{1}{2}I\omega^2$

$$=\frac{1}{2}mv^2\cdot\frac{mr^2+I}{mr^2}.$$

$$\therefore v^2=2gh\cdot\frac{mr^2}{mr^2+I}=\frac{5}{7}\text{ of }2gh,$$

and is therefore the same for all the spheres.

(35) In the case of cylinders, $v^2 = \frac{2}{3}$ of $2gh = \frac{4gh}{3}$. The *average* velocity will be half this final velocity, hence the time occupied in descending a slope s, whose height is h, is $s\sqrt{\frac{3}{gh}}$.

The time taken by a sphere would be $s\sqrt{\frac{14}{5gh}}$;

$\therefore$ the ratio of the times $= \sqrt{\frac{14}{15}}$.

(36) 560 foot-poundals, or about $17\frac{1}{2}$ foot-pounds.

(37) 140 foot-pounds.

EXAMPLES XVI. *Pages* 145–147.

(1) 10, $10\sqrt{2}$ or 14·142, $10\sqrt{3}$ or 17·32, $10\sqrt{(2+\sqrt{2})}$, $10\sqrt{(2+\sqrt{3})}$.

(2) Each equals $6\sqrt{2}$.

(3) $8\frac{1}{2}$ lb. weight (nearly).

(4) The fractions of the weight in each case are

$$\frac{\sqrt{3}-1}{\sqrt{2}};\ \frac{1}{\sqrt{3}};\ \frac{1}{\sqrt{2}};\ 1;\ \frac{\sqrt{3}+1}{\sqrt{2}}.$$

The best way with a single cord is to use two nails as far apart as the eyes on the picture.

(5) (1) $\frac{1}{\sqrt{3}}$ of 10 lb. weight, $=5·77$ lb. weight; normal pressure $= 11·55$ lb. weight. (2) 5 lb. weight; normal pressure $=$ 8·66 lb. weight.

(6) $\frac{1}{4}$ ton weight.

(7) (*a*) $200\sqrt{26}$ lb.; (*b*) 1000 lb.

(9) $\sqrt{247} = 15·7$. If OC be the resultant, in the diagram, of the given forces acting on the particle at O, and if a circle be drawn on OC as diameter, any pair of chords drawn through O at right angles to each other will be a pair of components equivalent to OC, and therefore to the given forces. Their magnitudes are given by the equation $x^2+y^2=247$.

(10) $5\sqrt{57} = 37\frac{3}{4}$ units.

(11) The forces can be reduced to $10\frac{1}{2}$ units acting along the line of action of the greatest of the given forces, and $\frac{3\sqrt{3}}{2}$ perpendicular to it, the resultant being $\sqrt{117}$. (*Or*, they may be

reduced to 9 units and 3 units acting respectively along the lines of action of the given forces 10 and 9.)

(12) 29·72.

(13) (*a*) 2500 lb. weight; (*b*) Depends upon a new set of the sail, and the wind speed, and too many assumptions to permit a satisfactory brief answer.

(14) Resultant = 25·58 units, 18·22 along AB, and 17·96 along DA.

(15) 77 lb. (nearly).

(16) Resultant = 24·2 lb. weight, its components along AB, AD being $15\frac{8}{13}$ and $18\frac{6}{13}$ respectively.

(17) $\frac{3}{4}$ second.

(18) $\frac{1}{3}\sqrt{3} = 0·577$.

(19) In No. 18, $56\sqrt{3} (= 97)$ lb. weight; in No. 17, 61·94 lb. weight.

EXAMPLES XVII. *Pages* 166-169.

(1) 5 and 15, or 30 and − 10.

(2) 3 inches from the middle.

(3) 2 inches from the middle.

(4) The diameter of the hole being $\frac{1}{4}$, its area is $\frac{1}{16}$ of that of the whole disc; so the centre of gravity is $\frac{1}{30}$th of the radius of the disc away from its centre, on the side opposite the hole.

(5) Magnitude of resultant, $2\sqrt{2}$ lb. weight. Its line of action lies outside the square 3 inches from the point where the two larger forces act, and is perpendicular to the diagonal through that point.

(6) 5 foot-pounds.

(7) 50 lb. weight.

(8) 86·6 and 100 lb. weight.

(9) A couple whose moment is 3 foot-pounds.

(11) $133\frac{1}{3}$ lb. weight; $166\frac{2}{3}$ lb. weight.

(12) Each equals 15 lb. weight.

(13) $27\frac{3}{11}$ and $32\frac{8}{11}$ lb.

(14) 4 feet.

(15) $86\frac{2}{3}$ and $293\frac{1}{3}$.

(16) 6·58 inches from the opposite corner, *i.e.* nearly $\frac{1}{2}$ inch from the middle.

(17) $\frac{1}{3}$ inch from centre of disc.

(18) $5\frac{1}{2}$ inches from the 1st side of the square, and $4\frac{1}{2}$ inches from the 2nd side.

(19) 7 inches from the 1st side of the square, and 5 inches from the 2nd side.
(20) On the diagonal joining the 3 and 6 lb. weights, 7 inches from the latter.
(22) 8 inches from the vertex.
(23) 7 inches from the heavier weight.
(24) 1 inch.
(25) $\frac{7}{16}$ inch from the middle.
(26) 3 inches from the loaded side.
(27) 21·6 inches from the shorter bar, and 9·6 inches from the longer.
(28) $13\frac{1}{4}$ inches from the top, and $10\frac{3}{4}$ from the bottom.
(29) 12·8 inches from the top, and 11·2 from the bottom.
(30) 1·282 inch from the centre, away from the hole.
(31) About $1\frac{5}{8}$ inch from the centre of the plate.
(32) $\frac{1}{16}$ inch from centre of plate.
(33) $\frac{1}{12}$.
(34) $\frac{5}{9}$ of height from the shortest side = 2·06 inches from that side.
(35) 2·18 inches, *i.e.* $\frac{10}{17}$ of the height, from the shortest side.
(36) $4\frac{4}{11}$ inches from the remaining short bar.
(37) $1\frac{1}{3}$ inch from the shortest side, and 1 inch from the medium side, in each case.

EXAMPLES XVIII. *Pages* 189–191.

(4) 10 lb. weight.
(5) $5\sqrt{2}$ lb. weight.
(7) At a height $\frac{1}{\sqrt{2}}$th of the length of the rope, because then the perpendicular distance of the rope from the base of the column will be greatest, and therefore the moment of any stress in the rope about it will be a maximum.
(8) 45°.
(9) 30°.
(10) 5 feet 4 inches.
(11) 10 lb. weight, at an angle of 30° below the horizontal.
(13) If $2l$ is the length of the bar, and a is the distance of the rail from the wall, the point of the rod which rests on the rail is at a distance x from its wall-end given by $x^3 = a^2 l$.
(14) 11·3 and 80·8 lb. weight.
(17) 5 ft. 8 in.; 54·7 and 41·4 lb. weight.

(18) It will fall over when the slope is 3 vertical to 8 horizontal.

(19) A horizontal force of (about) 7 lb. weight applied 6 inches up.

(20) $5\frac{2}{11}$ inches above the ground, *i.e.* $\frac{9}{11}$ of an inch below the centre of the sphere (which is the metacentre).

(21) Stable.

(22) $24\frac{1}{3}°$ (by protractor, approx.).

(23) 2 inches from the centre; 1 foot-pound. Considering the circular cross-section of the cylinder which contains G (the centre of gravity), G will lie somewhere on a concentric circle of 2 inches radius. The vertical line through the point of support will cut this circle in two points; G must be at one of these points—the upper will give unstable, and the lower, stable equilibrium.

(24) 1 vertical to 3 slant. In this case the vertical line through the point of support will just graze the above described gravity circle.

EXAMPLES XIX. *Pages* 200–202.

(3) 448 poundals, or a weight of 14 lb.

(4) $17\frac{1}{2}$ lb. weight.

(6) 1·486 feet a second (nearly).

(7) Acceleration of $W = \frac{1}{20}g$; acceleration of $P = \frac{1}{20}g$; time $= \frac{1}{2}\sqrt{29}$ seconds.

(8) $\frac{3}{7}$ of distance between hooks from middle hook.

(9) 3 tons.

(10) $10\frac{10}{17}$ lb. weight; $a = \frac{1}{17}g$, or nearly 2 ft./(sec.)^2.

(11) 39 cwt.; 9·908 lb.; 1·293 lb.

(12) *Pulleys;* 1 lb. each. *Tensions;* 7, 13, 25 lb. weight.

(13) *Pulleys;* 2 lb. each. *Tensions;* 20, 38, 74 lb. weight. *Pull on ceiling;* 132 lb. weight.

(14) *Pulleys;* 5 lb. each. *Tensions;* 20, 35, 65, 125 lb. weight.

(15) $4\frac{3}{7}$ lb. weight.

(16) Weight = 48 times the required effort. During each turn of the windlass the end of the handle (where the effort is applied) moves through 3π feet $= \frac{3 \times 22}{7}$ feet; and the weight is raised $\frac{1}{48}$ of this—viz. $\frac{22}{16 \times 7} = \frac{22}{112}$ of a foot. Therefore 112 turns are required to raise the weight 22 feet.

EXAMPLES XX. *Page* 210.

(1) The strain is 3 parts in a thousand; the stress is 600,000 grammes weight per square centimetre, hence the ratio of stress to strain, or Young's modulus, is 200 million grammes per square centim., or about 200,000 atmospheres linear traction per unit strain.

NOTE.—By an 'atmosphere' is meant a pressure of about 14·6 lb. to the square inch, or 1 ton to the square foot, or a kilogramme per square centimetre, or 30 inches of mercury, or 34 feet of water; or, when precise, 75 cm. of mercury at a certain temperature.

(2) 200 atmospheres, or a load of two kilogrammes.

(3) $\frac{1}{100}$ E traction.

(4) $\frac{1}{4000}$ E thrust.

(5) 1000 times 14·6 lb., or say $6\frac{1}{2}$ tons.

(6) One-fortieth per cent.

(7) Three-twentieths per cent.

(8) Three parts in two thousand for about 60 atmospheres, or ·000025 per atmosphere.

(9) The volume elasticity is nearly the reciprocal of the last answer—namely, 40,000 atmospheres; the Young's modulus is 60,000 atmospheres.

[To complete this it may be as well to state, without proof, that the reciprocal of the rigidity of the above material would be two-tenths plus two-fortieths per cent. for the 60 atmospheres, or that the rigidity itself would be 24,000 atmospheres.]

(10) 150 parts in 20,000, or about three-quarters of one per cent.

(11) The average compression would be that at the depth of 1 mile, found above. Hence the rise of level would be three parts out of each 400 in the two miles—namely, about 120 feet. [John Canton, 1747.]

(12) 1 atmosphere.

(13) 1·4 atmosphere.

EXAMPLES XXI. *Pages* 222, 223.

(1) The mechanical advantage of the lever is 60, and of the press itself 256; hence, the total mechanical advantage is 15,360, and the greatest weight the man can raise is 1440 tons.

(2) 13,600 grammes weight on the bottom, and 6800 on each side.

(3) 2075 grammes weight, made up as follows : (1) 125 grammes weight on the upper half of the side, due to the water, acting at a point $\frac{2}{3}$ down that half, *i.e.* $3\frac{1}{3}$ cm. from the top; (2) a transmitted pressure of 250 grammes weight on the lower half, due to the water, and acting at the centre of that half, *i.e.* $7\frac{1}{2}$ cm. from the top; (3) $13\cdot6 \times 125$ grammes $= 1700$ grammes weight on the lower half, due to the mercury, acting $\frac{2}{3}$ down that side, *i.e.* $8\frac{1}{3}$ cm. from the top. These make a total pressure of 2075 grammes weight acting at 2·07 cm. from the bottom.

(4) 30 'atmospheres' or about 440 lb. weight per square inch.

(5) π *tonnes*, or 3141·6 kilogrammes weight.

[The atmospheric pressure would hold them together about ten times as strongly if the interior were exhausted.]

(6) 14 tons (nearly); $3\frac{1}{3}$ tons (nearly).

(7) Downward pressure on base $= 250$ lb. weight.
Pressure on each side $= 140\sqrt{5}$ lb. weight, of which 140 lb. weight is the vertical component. Weight of wedge in order to float $= 280 - 250 = 30$ lb.

(8) 124 tons weight, acting horizontally $8\frac{1}{2}$ feet from bottom of gate.

(9) 6·154 lb. weight; 6 inches down.

(10) 3·077 lb. weight; 4 inches down.

(11) $74\frac{3}{4}$ lb. weight.

(12) 1000 lb. weight.

(13) $2\frac{1}{3}$.

(14) (*a*) $\frac{336}{\pi}$ or about 107 lb. weight per sq. centimetre; (*b*) $11\frac{1}{4}$ tons weight; $\frac{10}{7}(20)^2 = 571\frac{3}{7}$.

(15) (*a*) $1543\frac{1}{4}$ foot-pounds; (*b*) $\frac{9}{16}$ of (*a*) $= 868$ foot-pounds.

EXAMPLES XXII. *Pages* 238-241.

(1) $\frac{4}{3}\pi$ *tonnes*, or 4188·8 kilogrammes weight.

(2) 1500 grammes.

(3) 816 cubic centimetres.

(4) 0·7.

(5) 0·8421 and $1\cdot0\dot{6}$; *i.e.* $\frac{16}{19}$ and $\frac{16}{15}$.

(6) To the division 100.

(7) 1·2.

(8) 9·5 grammes.

(9) 5.

(10) 0·6 and 3 respectively.
(11) 1·8.
(12) 3·2.
(13) $\frac{1}{8}$.
(14) $55\frac{1}{2}$ cubic inches (nearly); sp. gr. = 5.
(15) $\frac{2}{3}$.
(16) 2·05 approx.
(17) 1·6.
(18) $\frac{7}{8}$.
(19) 50 parts of each.
(20) 14 inches; $\frac{9}{7}$.
(21) (1) 10 : − 44; (2) 168 : 167.
(22) 1·1; (2) $1\frac{3}{8}$; (3) $1\frac{2}{5}$.
(23) 0·7.
(24) 7 : 6.
(25) $2\frac{1}{2}$ and $\frac{3}{4}$.
(26) 0·113 millimetre; 8 grammes.
(27) 250 cubic centimetres.
(28) $6\frac{3}{19}$ and $\frac{18}{19}$.
(29) Volume = $14\frac{2}{3}$ cubic centim.; diameter = 3·02 centim.
(30) $209\frac{5}{11}$ grammes.

EXAMPLES XXIII. *Pages* 255–257.

(1) 75 centimetres.
(2) 13·6 inches.
(5) 3 megadynes per square centimetre.
(6) About 12 tons weight.
(7) 80·517 grammes.
(9) 9·523 grammes.
(11) 14·2.
(12) 11·3 metres.
(13) 1334.
(14) $74\frac{2}{17}$ inches.
(15) 1019·7 grammes weight, = 1,000,325 dynes.
(16) 1354·7 grammes weight per square centimetre.

EXAMPLES XXIV. *Pages* 264–266.

(1) 17 feet.
(3) 74 centimetres.

(4) After one stroke, 70 centimetres; after two, $63{\cdot}\dot{6}\dot{3}$; after three strokes, $57{\cdot}\dot{9}42\dot{0}$, and so on; each time dividing by 11 and multiplying by 10.

(5) ·652 ton weight per square inch. At a depth of 3366 feet.

(6) (Neglecting the thickness of its wall, and taking the barometric height as equal to 34 feet of the water), 0.

(7) 15 to 17.

(8) 1005 grammes weight.

(9) 10.

(10) It is reduced to $\frac{9}{10}$ by 1 stroke, and to $\frac{81}{100}$ in 2 strokes.

(11) 0·463 of its initial pressure.

(12) $29\frac{4}{11}$.

(13) 38 ft., or 40 ft., according as barometer is taken as 34 or 32 ft.

(14) 9 feet, leaving only 1 foot depth of air space.

(15) 500 cubic feet.

(16) $66\frac{2}{3}$ inches of mercury.

(17) $1\frac{35}{72}$ atmosphere.

(18) 30 inches.

(19) Mercury has risen $1\frac{1}{4}$ cm.; *i.e.* lip is submerged $6\frac{1}{4}$ cm.

ANSWERS TO MISCELLANEOUS EXERCISES.

SET I. *Pages* 267, 268.

[*N.B.*—Air-resistance neglected throughout.]

(1) 40,000 feet ; 100 seconds.

(2) 5 seconds after the last ball is thrown, at a height of 560 feet.

(3) 2 seconds.

(4) 640 feet below the balloon ; *velocity*, 224 ft./sec. downwards.

(5) 52 ft./sec.

(6) In 10 seconds ; 240 ft./sec.

(7) In $\frac{1}{4}\sqrt{10}$ or about 0·8 second.

(8) In $2\frac{1}{2}$ seconds, 100 feet from the top.

(9) 240 poundals, or $7\frac{1}{2}$ lb. weight ; $1\frac{1}{2}$ second.

(10) Vertically as seen by the passengers ; in a portion of a parabola whose vertical and horizontal dimensions are about 9 feet and 57 feet respectively, relatively to the earth.

(11) It goes up 16 feet, describing a parabola so as to be always vertically over the man's hand, to which it returns in 2 seconds, having really travelled a horizontal distance of about 150 feet.

(12) $7\frac{1}{2}$ seconds.

(13) 47 cwt. ; 180 feet.

SET II. *Page* 268.

(1) Magnitude $= \sqrt{31} = 5{\cdot}57$. Equivalent to 0, 0, 0, 0, 1, 5 in the given direction.

(2) 26 along line of greatest force.

(3) 19 lb. weight in the long cord ; 107·3 in the short cord.

(4) 7·3 feet and 11·6 feet.

(5) 99·4 and 54·6 lb.

(6) 10 units, at 4·6 inches from first force.
(7) $8\frac{1}{2}$ feet.
(8) $7\frac{1}{8}$ inches from the middle of the pole.

Set III. *Pages* 268, 269.

(7) 75 centimetres.

Set IV. *Pages* 269, 270.

(3) 15·94 oz. weight ; 2 ft./sec. per second.
(5) $3\frac{3}{8}$ inches from the side containing the first two weights, and equidistant from these weights.
(7) 933,912 dynes per square cm.—*i.e.*, ·934 atmosphere.
(9) *Horizontal pressure*, $10\frac{7}{8}$; *vertical*, 56 ; *ground resultant*, *pressure*, about 57.
(10) 576 feet ; 12 seconds ; 432 feet up.
(11) 40 ft./sec. per second ; 500 feet.
(12) 2·15 lb. weight.

Set V. *Pages* 270–272.

(2) $\sqrt{2}$ times as long.
(3) $210\frac{1}{4}$ feet ; $3\frac{5}{8}$ seconds.
(4) $4\pi^2 mr \div T^2$.
(6) $\frac{1}{2}\sqrt{15}$ (*i.e.* nearly 2) seconds.
(7) $\frac{2}{3}$ of the length of the rod from the axis.
(8) The vertical through the weight must pass through the intersection of the lines along which the unloaded rods lie.
(9) It must make an angle with the plane equal to the angle of repose.
(10) 16 inches ; $\frac{1}{4}$.
(11) 7 ; 72 grammes.
(13) $3\cdot\dot{1}\dot{8}$ kilogrammes.
(14) If μ is the coefficient of friction with the ground, and μ' with the wall, and if the centre of gravity is $\frac{1}{n}$ of its length from the ground, then, if h is the elevation of the top of the ladder, and b is the distance of its foot from the wall,

$$\frac{h}{b}=\frac{1}{n\mu}-\mu'\left(1-\frac{1}{n}\right).$$

(15) $a \div \mu\sqrt{3}$, where a is the length of an edge of the pyramid, and μ is the coefficient of friction. (The drawing cord must be fastened to an *edge*, to give this best chance of not upsetting.)

(16) v must be greater than $\sqrt{(gr)}$.

SET VI. *Pages* 272, 273.

(3) In $3\frac{1}{4}$ seconds; 3250 feet.

(5) $4\frac{4}{5}$ ft./sec. per second; 60 feet.

(6) Tension $= 10\frac{10}{17}$ lb. weight; acceleration $= \frac{1}{17}g$.

(8) The recoil velocity is $\frac{3}{4}$ of the approach velocity. See also answer to IX. (13).

(9) The required moment exceeds the central moment by the product of the mass into the square of the distance between the axes.

(10) If a is the distance of the centre of rod from the axis of suspension, and x the length of the equivalent simple pendulum, $a(x-a)=\frac{1}{12}l^2$; x is a maximum, and the time of swing is therefore a minimum, when $a^2=\frac{1}{12}l^2$.

(11) One five-hundredth of an inch.

(12) ii. g varies apparently on account of centrifugal force, and really on account of the shape of the earth. The two effects add together.

(13) iii. By equating the weight of a satellite or planet to its centrifugal force.

(14) Directly as distance from centre. About 21 minutes. The motion is simply harmonic and independent of amplitude. The maximum velocity at centre is $\sqrt{(g\text{R})}$, and g varies directly with R if density is given. Hence linear size $\div$ velocity is constant.

(16) The rod has $\frac{1}{n}$th immersed, such that $n+\frac{1}{n}=\frac{2}{s}$; or if water is above pivot, such that $n^2s=1$; otherwise the rod is vertical.

SET VII. *Page* 274.

(1) 196 feet; 112 feet per second.

(2) $2\frac{1}{2}$ secs.; 2500 feet from base of tower.

(3) $\frac{27}{32}$ of 3 cwt. ; or 283½ lb. weight.

(4) $1 \cdot 0\dot{6}$ foot, $2 \cdot 1\dot{3}$ feet per sec. 22·4 lb. weight.

(5) 1664 feet per sec.

(6) 24·7 revolutions per sec.

(7) Speeds proportional to the chords. Times all the same.

(9) 30·73 foot-second units.

(10) 31,260 feet ; 44·2 seconds ; 7812·5 feet.

(11) About 80 lb. per inch of width.

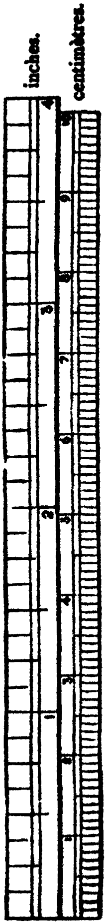
APPENDIX.

I. *Omission from Sect.* 103 (*page* 126).

Thus, in the case of a bent bow, the force with which the arrow has been pulled has varied from zero to a maximum, of perhaps 56 lb. weight. In that case, since the force is proportional to the strain (see sect. 158, Hooke's Law), the average force was 28 lb. weight, and so if the range of pull is 1 foot, the amount of energy stored up in the elastic strained wood of the bow is the equivalent of 28 foot-pounds.

This energy is imparted to the arrow when the string is released, and accordingly it flies off with 28×32 F.P.S. units of energy. If its mass is 4 ounces, this means an initial velocity of 84·6 feet per second. A very high initial velocity can be obtained by the use of a light straw needle-pointed arrow and a tapered bow of light stiff wood, like deal. Such an arrangement is used by Mr Boys for the purpose of drawing out molten quartz with extreme rapidity into excessively fine fibres.

C.G.S. SYSTEM OF UNITS.



COMPARISON OF CENTIMÈTRES WITH ENGLISH INCHES.

The C.G.S. system of units, as now adopted throughout Europe, is as follows:

The *centimetre* is the unit of length. A second is the unit of time. The cubic centimetre is the unit of volume.

The mass of one cubic centimetre of distilled water at its temperature of maximum density is called a *gramme*, and is the unit of mass.

(For the derived units *dyne* and *erg*, see sects. 46 and 86.)

The weight of a pound	=	445,000 dynes (approx.).
An atmosphere pressure	=	10^6 dynes per sq. centim.
	=	75 centims. of mercury.
1 foot	=	30·48 centimetres.
1 cubic inch	=	16·387 cubic centims.
1 pound	=	453·59 grammes.
1 gramme	=	15·43 grains.
The weight of a gramme	=	981 dynes (approx. in these latitudes).
1 centimetre	=	·3937 inch.
1 metre	=	3·281 feet.
1 litre	=	1000 cubic centims.

A velocity of one mile per hour = 44·704 centimetres per second.

The weight of one grain = 63·57 dynes.

A pressure of one pound-weight per square foot = 479 dynes per square centimetre.

An acceleration of 32·18 feet-per-second per second = 981 centimetres-per-second per second.

33 centimetres = 13 inches very nearly indeed.

For further details, see Professor Everett's book, *Units and Physical Constants.*

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